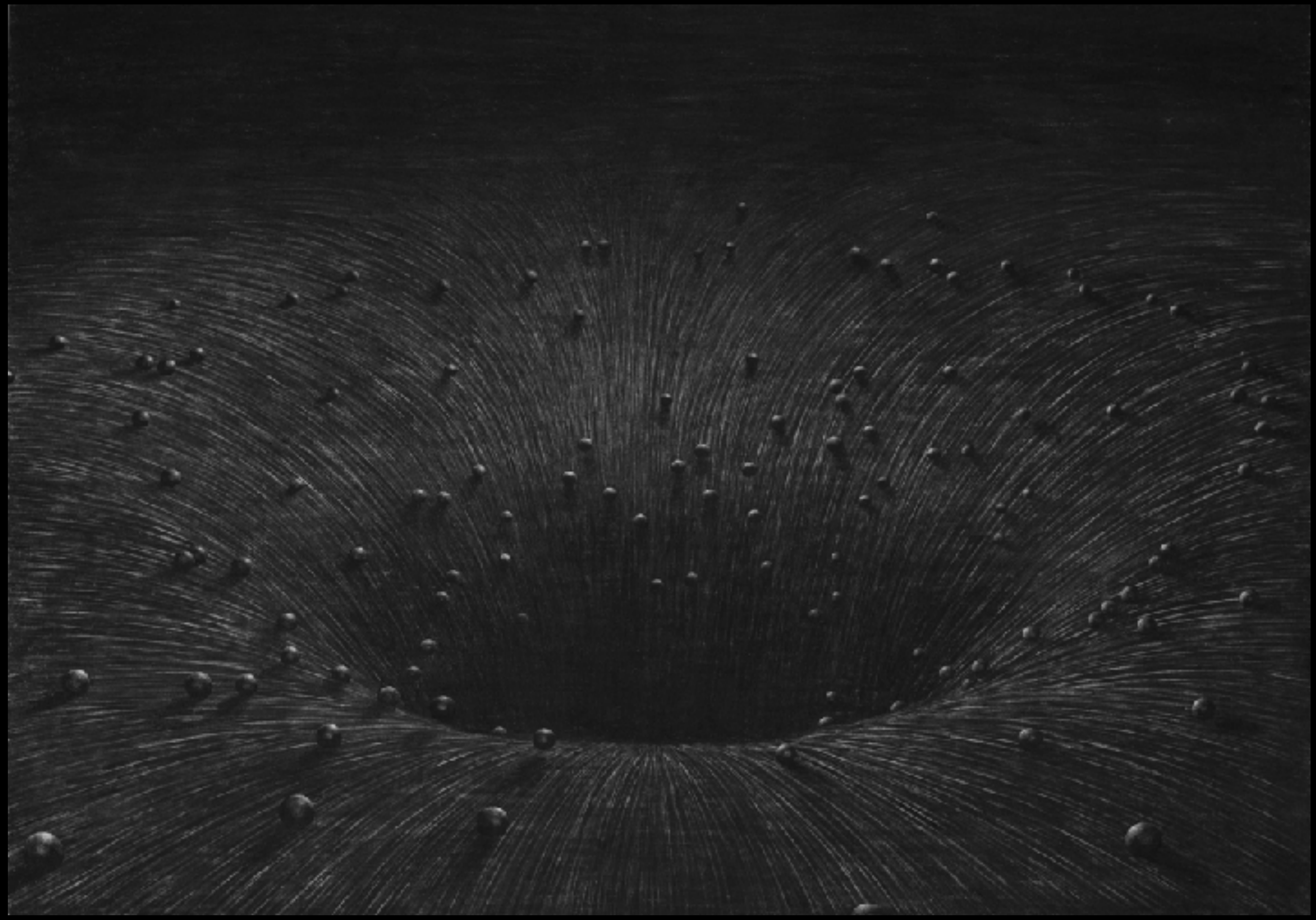
Nonlinear gravitational waves



Hole (2013), Levi van Veluw

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Danmarks Grundforskningsfond
Danish National Research Foundation

Outline

1. Linear and nonlinear gravitons
2. Propagation of gravitational waves
3. Nonlinearities in the waveform
4. Quadratic quasinormal modes and beyond
5. Gravitational waves and matter
6. Take-aways

I. Gravitons

Atmosphere (2015), Levi van Veluw

What is a photon?

The electromagnetic field strength has a (classical) decomposition…

$$2F_{ab} = \Phi_{AB}\bar{\epsilon}_{A'B'} + \mathbf{c.\,c.}$$

…in terms of which Maxwell equations are linear

$$\Box\Phi_{AB} = \Psi_{ABCD}\Phi^{CD} \overset{\text{Flat}}{=} 0$$

A photon is a **complex**, **positive-frequency**, freely propagating mode.

What is a photon?

A photon carries field strength.

Two photons carry a different field strength than a single photon.

There is no 3-photon vertex (without weak interactions)

What is a graviton?

The gravitational field (curvature) has a (classical) decomposition…

$$C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \mathbf{c.\,c.}$$

…Einstein equations are, however, nonlinear:

$$\Box\Psi_{ABCD} = 3\Psi^{EF}_{(AB}\Psi_{CD)EF}$$

A graviton is a **perturbative** **complex**, **positive-frequency**, freely propagating mode.

What is a graviton?

A graviton does not tilt the light-cones of Minkowski

Only after summing an infinite amount of gravitons do the light-cones change!

$$g_{ab}^{\mathrm{Sch}} = \eta_{ab} + \sum_{i=1}^{\infty} G^i h_{ab}^{(i)}$$

P.Damgaard & K.Lee (2024)

Non-trivial 3-graviton vertex!

What is a graviton?

"However, it is at this point that I wish to take issue with the standard view. I cannot (now) believe that a physical graviton should be described by linear gravitational theory. [...] if Einstein's theory is physically appropriate, then surely each graviton itself carries its measure of curvature. I cannot believe that curvature emerges only after an infinite sum has been performed."

The nonlinear graviton, R. Penrose (1976)

II. Gravitational Waves

Water (2013), Levi van Veluw

Propagation of gravitational waves

A gravitational wave does not propagate unchanged

$$g_{ab} = \eta_{ab} + h_{ab}$$

$$h_{ab} = h_{ab}^{(1)} + h_{ab}^{(2)} + \dots$$

$$\Box h_{ab}^{(1)} = 0$$

$$\Box h_{ab}^{(2)} = -\delta^{(2)} G_{ab}[h^{(1)}, h^{(1)}]$$

Isaacson (1967)
Mendonca & Cardoso (2002)
Kocsis & Loeb (2007)

Should we worry about this?

In other words, do the GWs that we measure on Earth have anything to do with the GWs emitted by a far-away source?

general relativity is a nonlinear theory. Therefore, gravitational waves will change as they propagate, due to nonlinear effects. As a consequence, the gravitational waves measured here on Earth might be very different than what was emitted by a source located very far away. Should we worry about this?

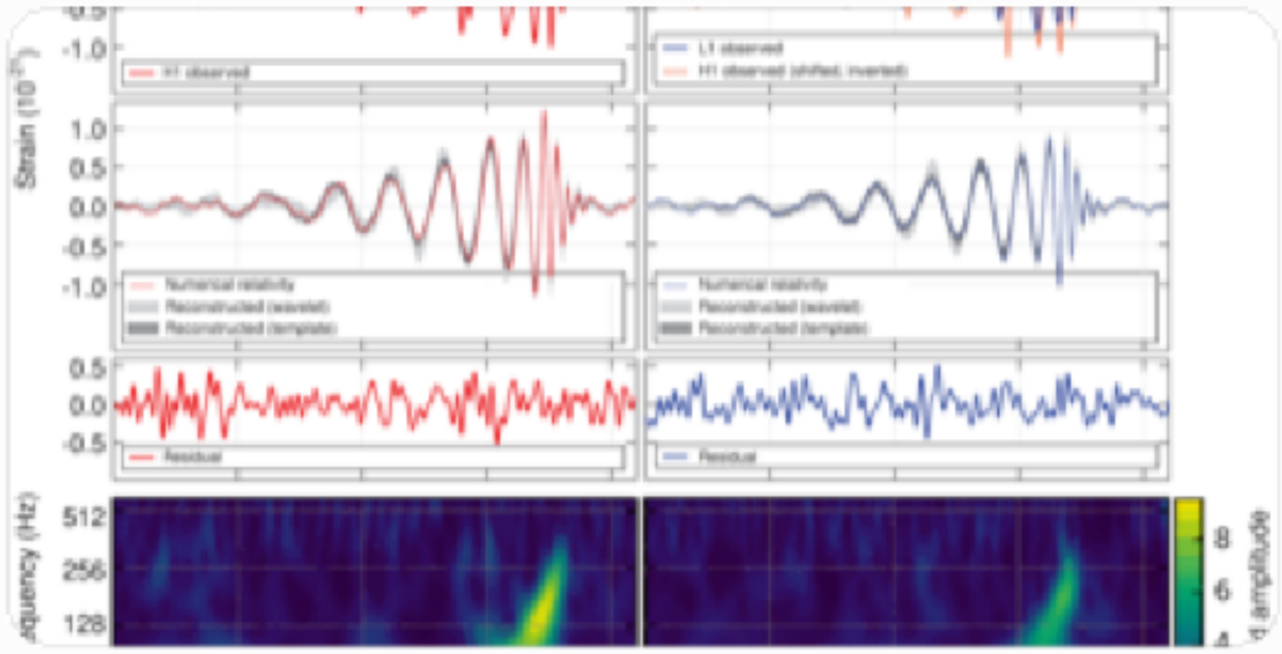
Answer via Dia

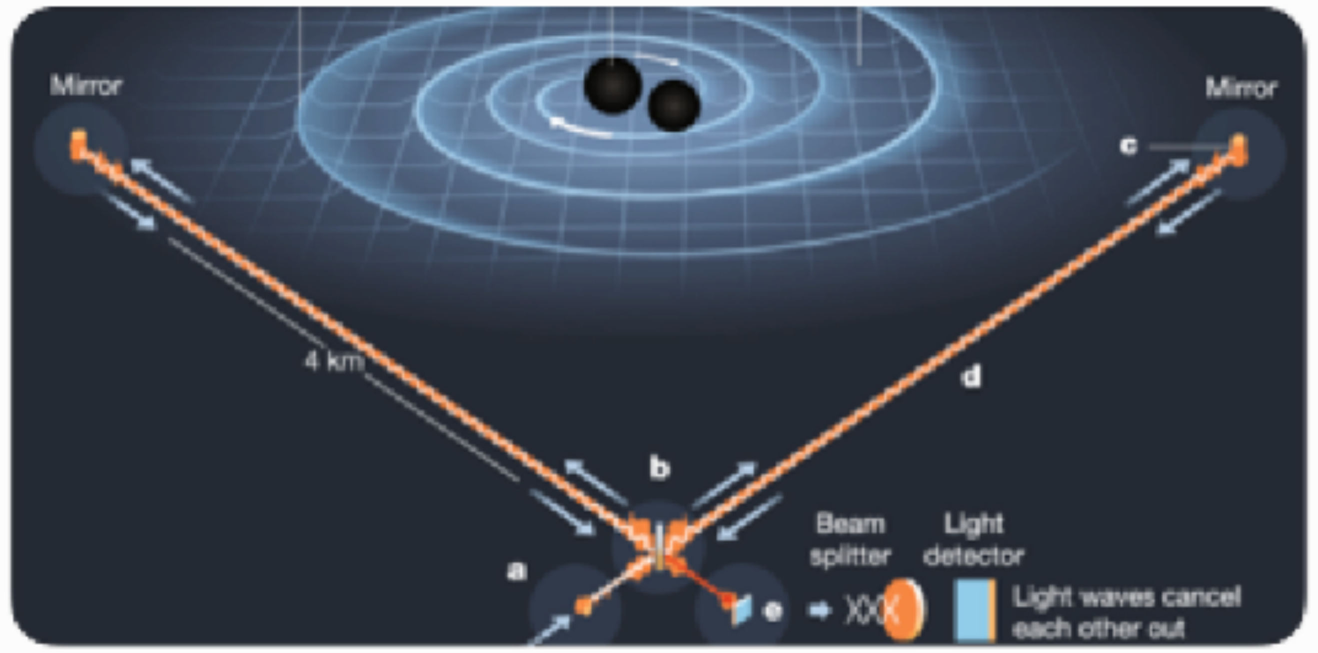
You're absolutely right that general relativity is a nonlinear theory, and that this nonlinearity means gravitational waves can, in principle, interact with themselves and with the curvature of spacetime as they travel. This raises a natural concern: could these nonlinear effects significantly distort the waves by the time they reach Earth, making it hard to infer what was originally emitted?

Should we worry about this?

So, should we worry?

No, nonlinear effects during propagation are negligible for gravitational waves detected on Earth.



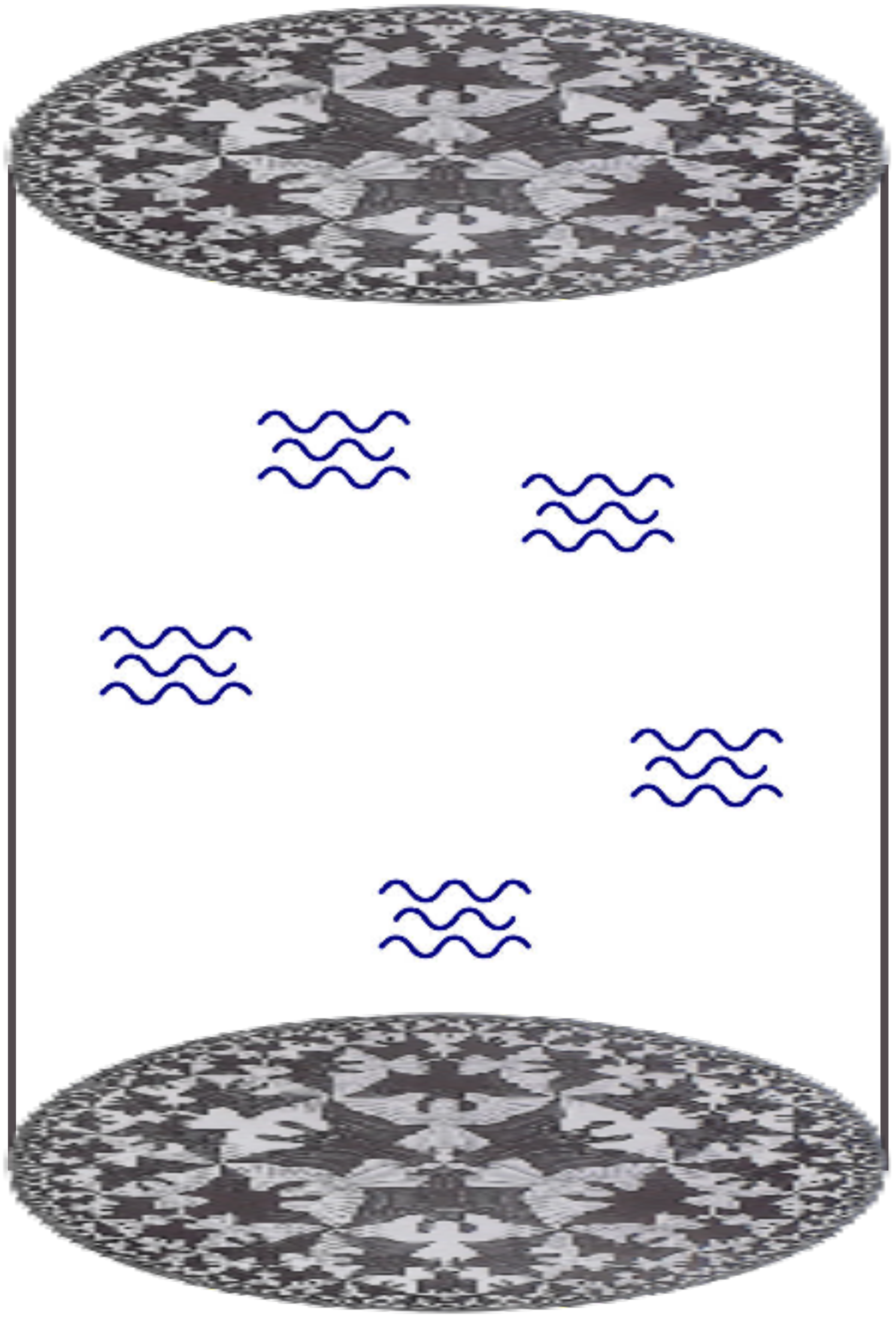


While it's theoretically possible for nonlinearities to affect wave propagation, in practice, the universe is kind: the waves are weak, space is mostly empty, and our models are good. That said, in extreme environments—like waves passing through dense matter or near other massive objects—nonlinear propagation effects could become more relevant, and researchers are actively exploring those edge cases.

Would you like to dive deeper into how numerical relativity or waveform modeling works?

Propagation of gravitational waves

How about in anti—de Sitter?



$$\Box\Phi^{(i)}_{\ell m} + V_{\ell m}\Phi^{(i)}_{\ell m} = \mathcal{S}^{(i)}_{\ell m}[\Phi^{(i-1)}, \ldots, \Phi^{(1)}]$$

$$\Phi^{(1)}_{\ell m} = \cos(\omega_{\ell,n} t) f_{\ell,n}(r)\,, \qquad L\omega_{\ell,n} = 1 + \ell + 2n$$

$$\Phi^{(3)}_{\ell m} \sim t\sin(\omega_{\ell,n} t) + \ldots$$

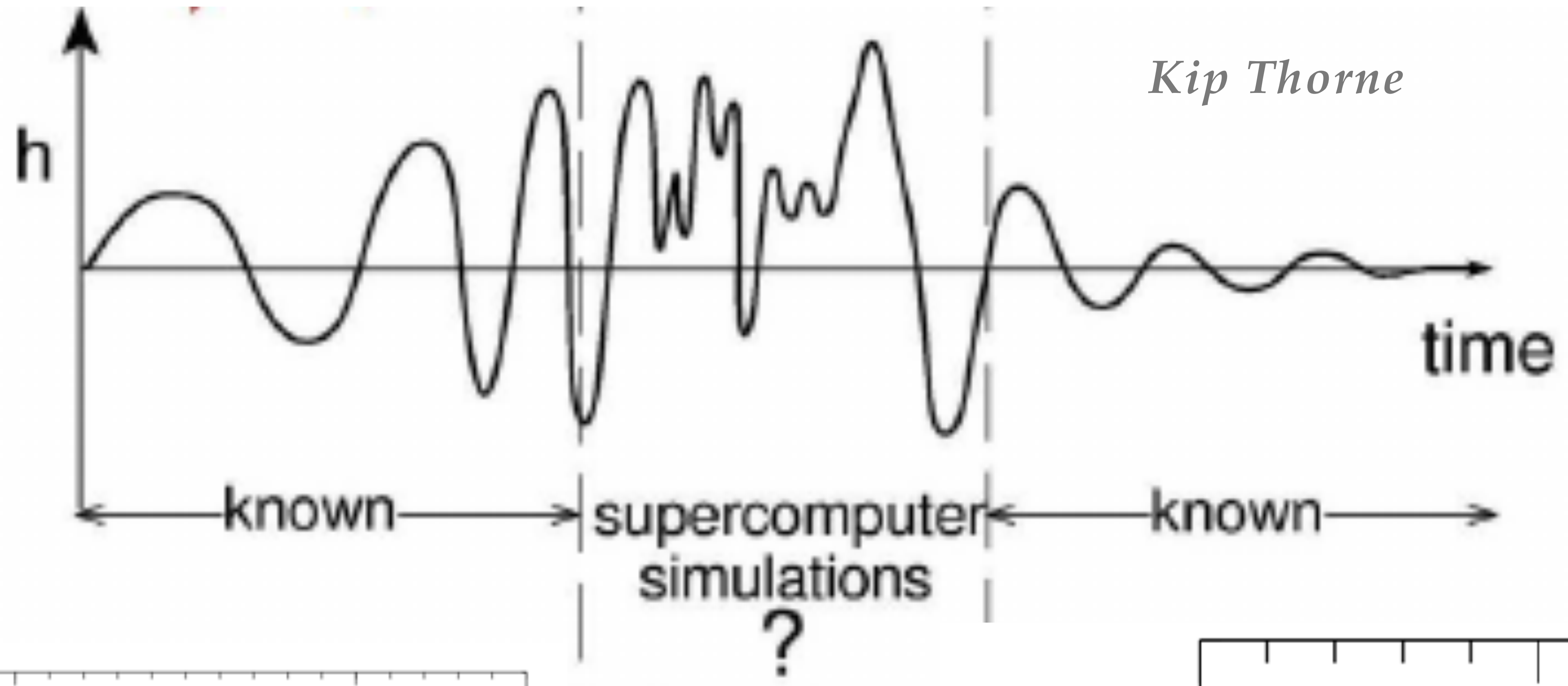
Resonant “4-graviton vertex”

Bizon,Rostworoski+ (2010+)
Dias, Horowitz, Santos (2011+)

III. Waveforms

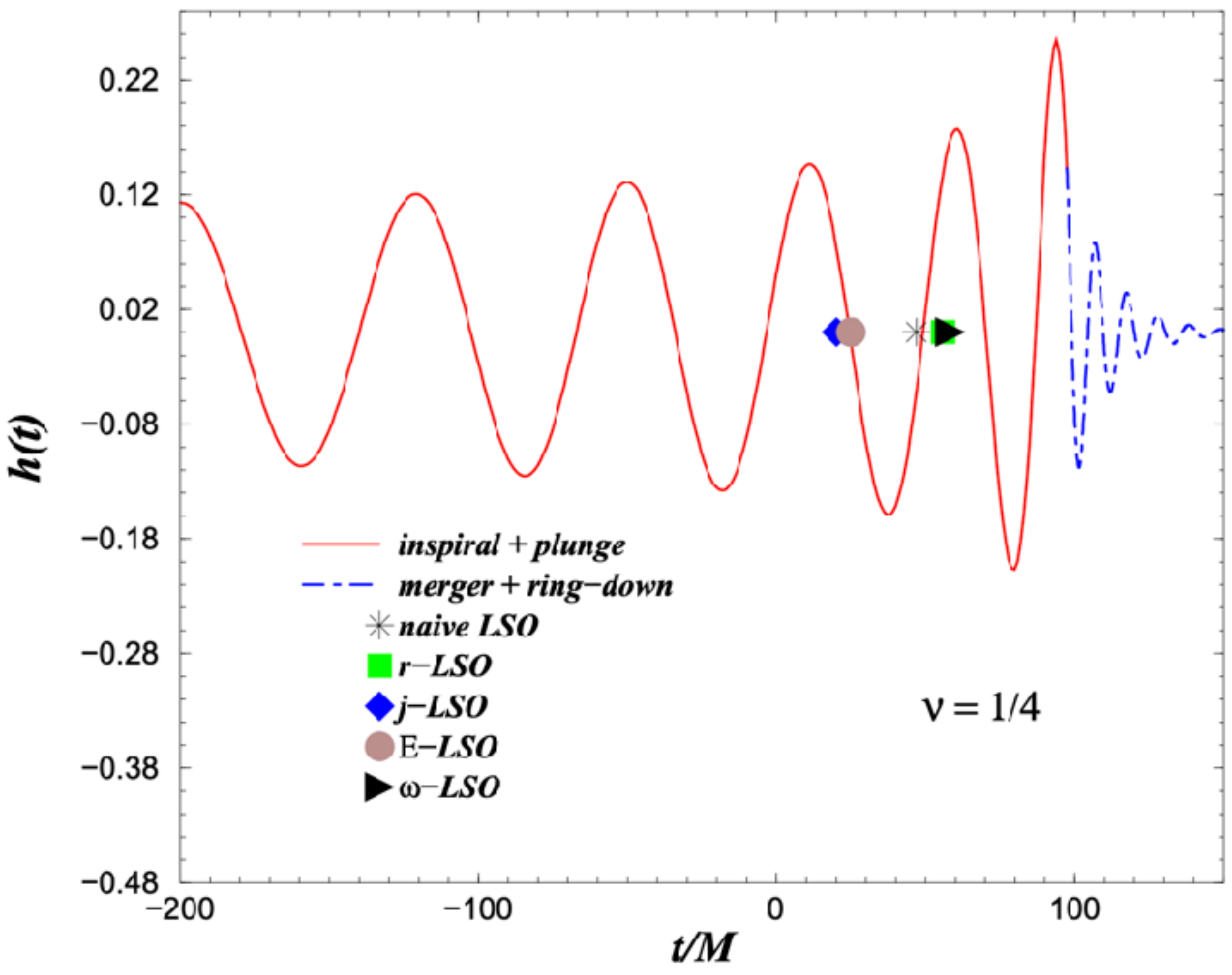
Shallow Hole (2013), Levi van Veluw

Merger waveforms

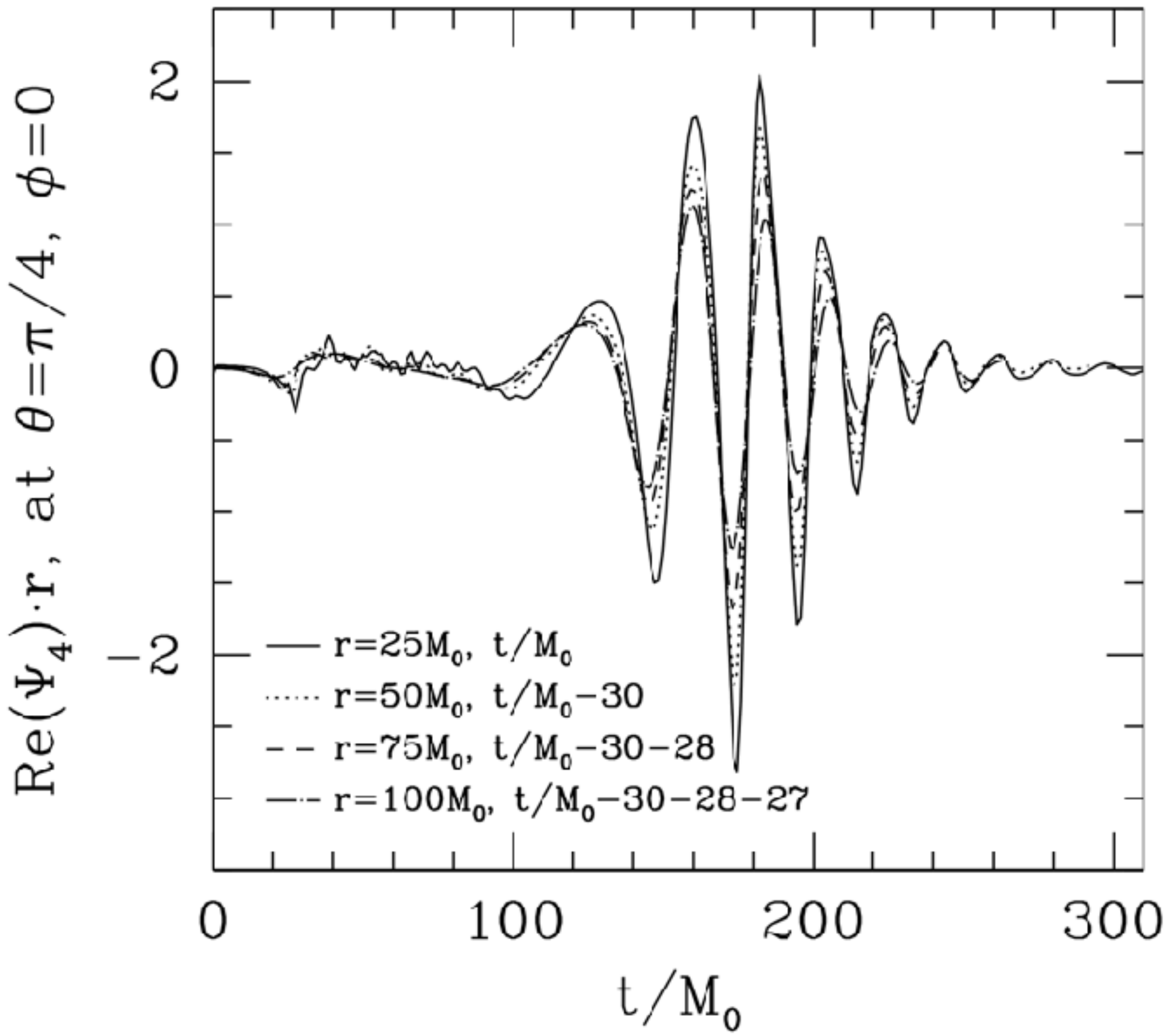


Kip Thorne

Buonanno&Damour (2000)



Pretorius (2005)



Why so simple?

The merger waveform is (surprisingly) simple.
This allows, e.g., matching perturbative solutions to the 2 body problem (pN/pM/EOB) with perturbations of a single BH (ringdown), in a smooth way.
Why?

Is it that simple?

Why not?

“Vanilla” mergers

Horizon absorption

Four dimensions

A challenge in waveform modelling

Higher order terms increase accuracy

No equivalent scheme in the opposite direction!

Fissioning white hole

A time-reversed black hole merger describes a white hole fissioning

— Retarded vs advanced solutions. Do we learn anything?

— Caustics at bifurcation point?

— Do higher orders in perturbation theory converge towards The nonlinear solution?

Husa & Winicour (1998)
Gomez + (2002)

IV. Black Holes

Hole (2013), Levi van Veluw

Black Hole Perturbation Theory

Let us start from the projection of Penrose's curvature wave equation

$$\mathcal{O}_4\Psi_4 + \mathcal{O}_3\Psi_3 + \mathcal{O}_2\Psi_2 = 0$$

Stewart & Walker (1974)

Consider a perturbative ansatz around the (algebraically special) Kerr background

$$g_{ab} = g_{ab}^{\mathrm{Kerr}} + \sum_{i=1}^{\infty} \epsilon^i h_{ab}^{(i)}$$

To linear order $\mathcal{O}_4\Psi_4^{(1)} = 0$

Teukolsky (1973)

Metric must be reconstructed at each order!

Beyond $\mathcal{O}_4\Psi_4^{(2)} = -\mathcal{O}_2^{(2)}\Psi_2^{(0)} - \mathcal{O}_3^{(1)}\Psi_3^{(1)} - \mathcal{O}_4^{(1)}\Psi_4^{(1)}$

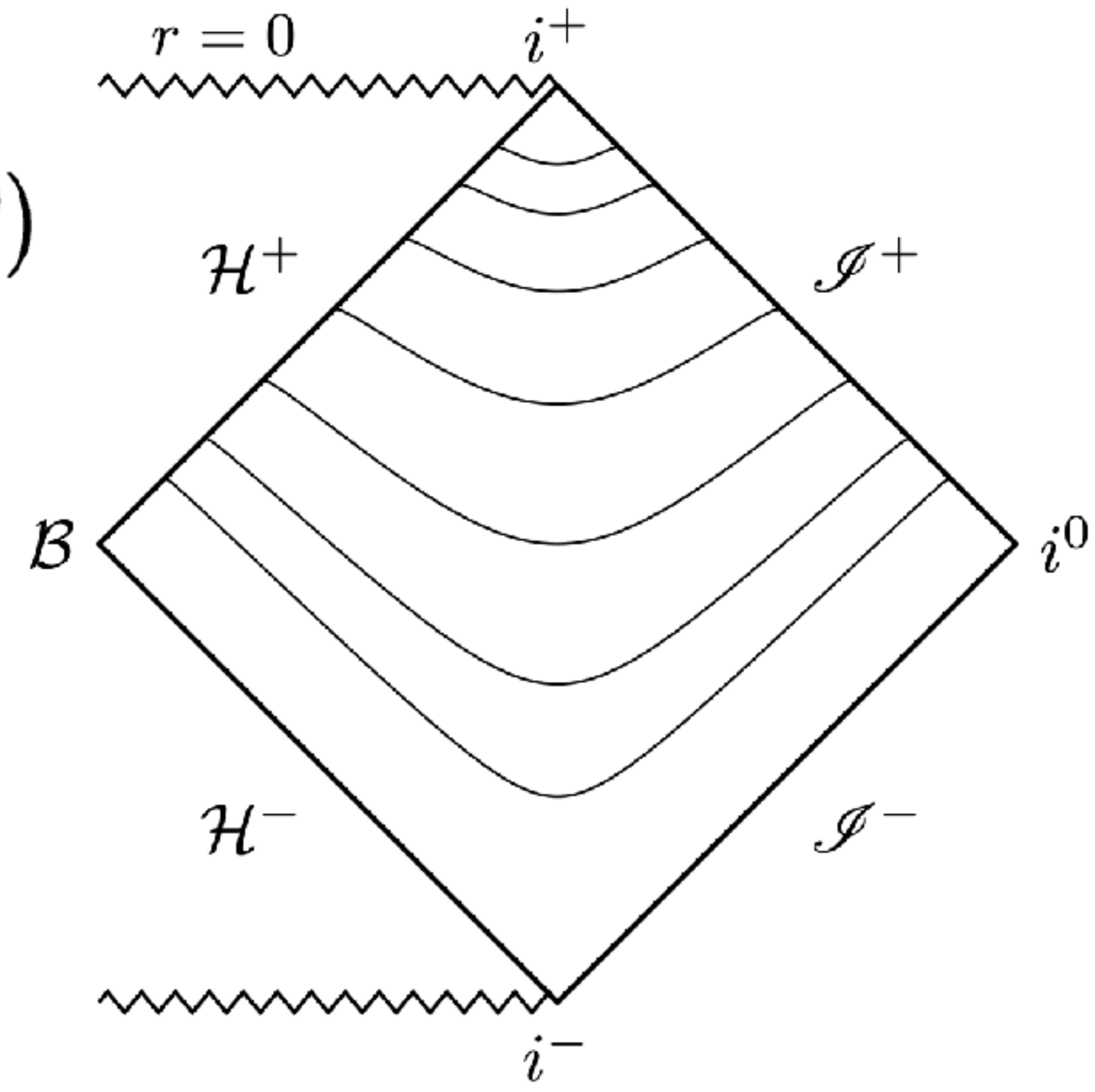
Campanelli, Lousto (1998)

Black Hole Perturbation Theory

Quasinormal modes are the quasi-particle states of Schwarzschild

$$\mathcal{O}_4 \Psi_4^{(1)} = 0$$

$$\Psi_4^{(1)} = \sum_{\ell,m,n,\pm} \mathcal{A}_{\ell mn\pm} e^{-i(\omega_{\ell mn\pm} t - m\phi)} f_{\ell mn\pm}(r) S_{\ell mn\pm}(\theta)$$



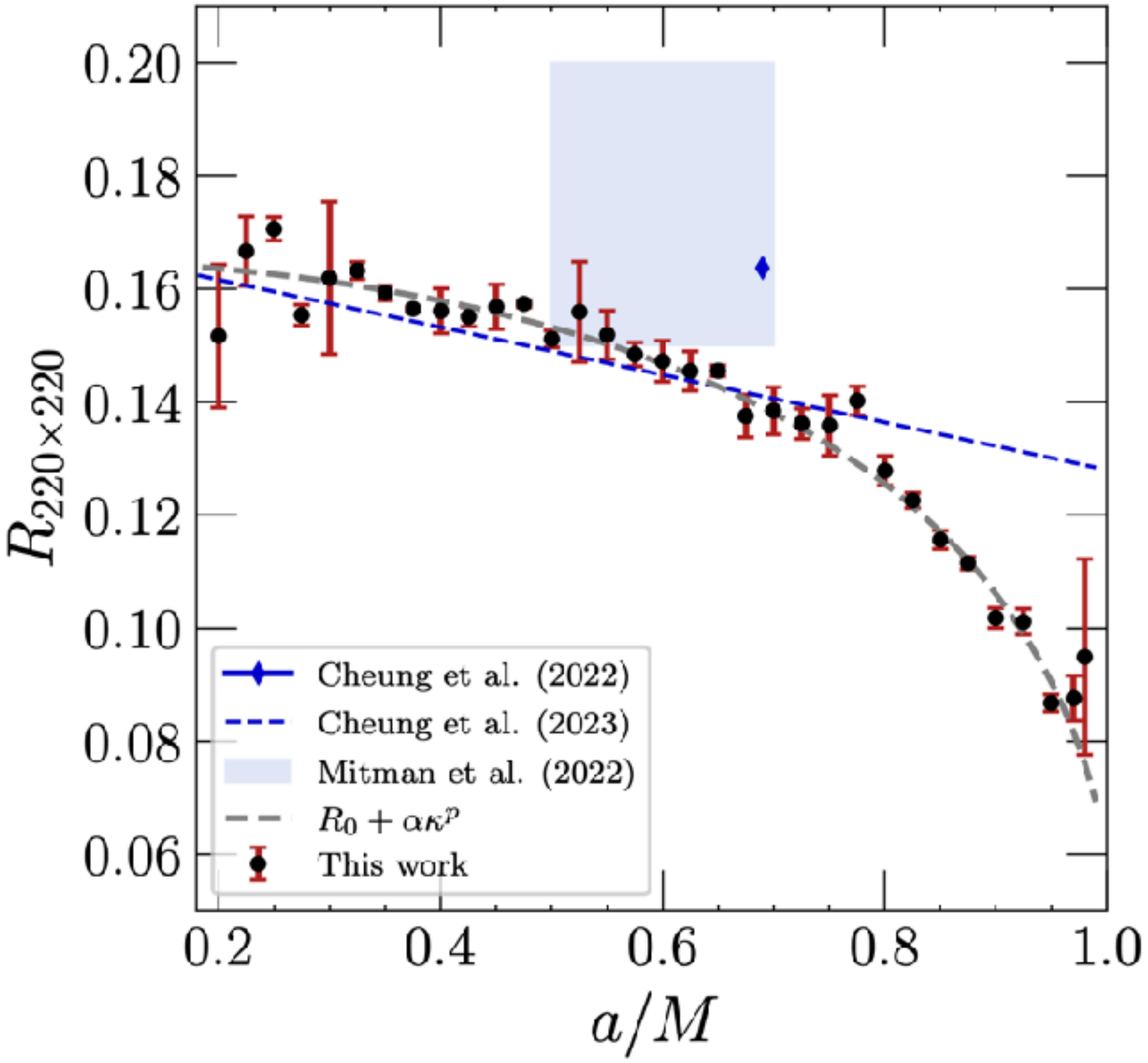
	AdS	Kerr
Real?	YES	NO
Commensurate?	YES	NO
Degenerate?	YES	YES

Quadratic Quasinormal Modes

Higher order effects oscillate at frequencies which are not poles of the propagator

$$\mathcal{O}_4 \Psi_4^{(2)} = e^{-2i\omega t} \times [\ldots]$$

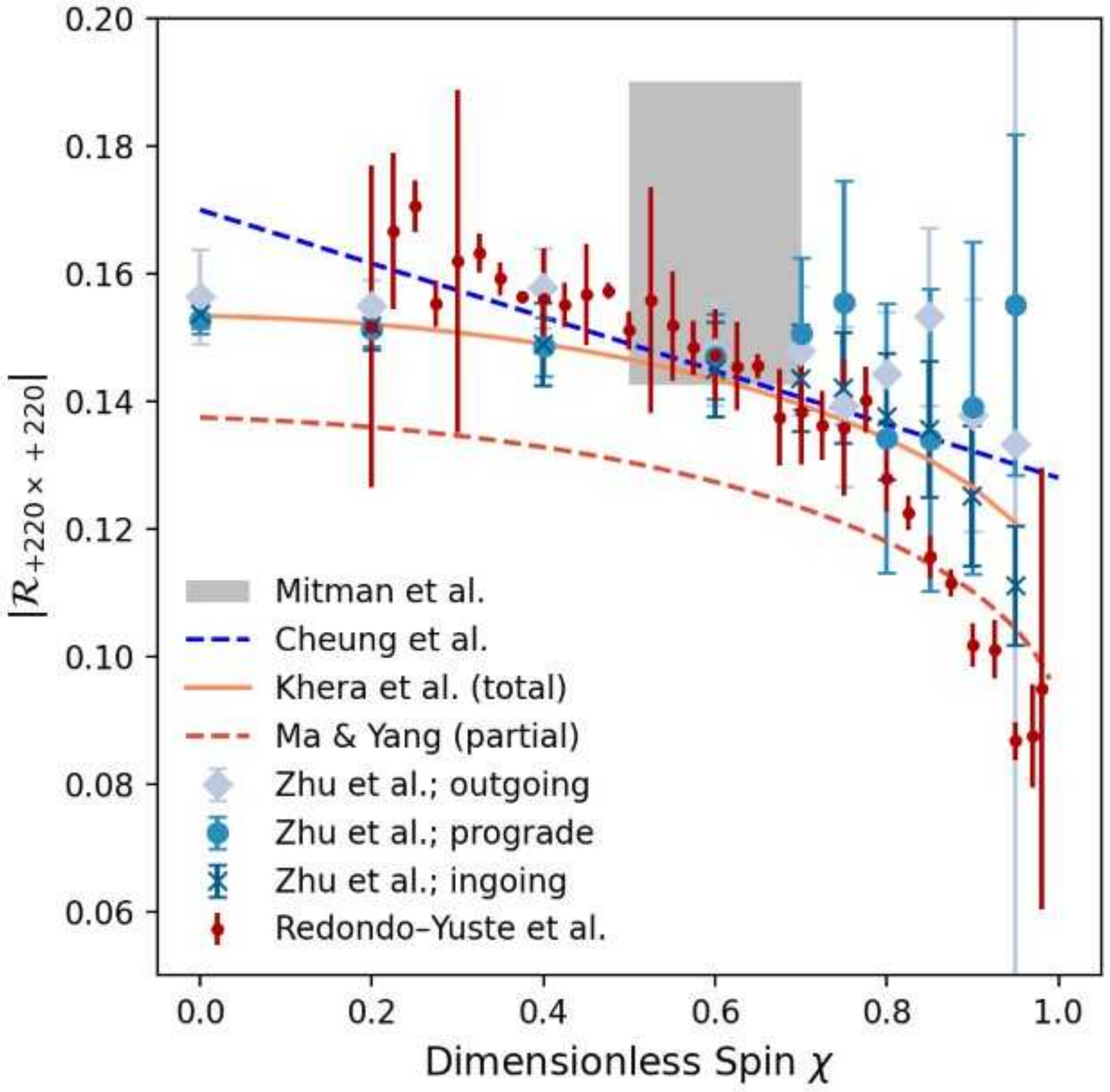
$$(\Psi_4^{(2)})_{\ell m} = \mathcal{R}^{\ell m}_{(\ell_1 m_1 n_1 \sigma_1)\times(\ell_2 m_2 n_2 \sigma_2)} \mathcal{A}_{(\ell_1 m_1 n_1 \sigma_1)} \mathcal{A}_{(\ell_2 m_2 n_2 \sigma_2)} e^{-i(\omega_{\ell_1 m_1 n_1 \sigma_1} + \omega_{\ell_2 m_2 n_2 \sigma_2})t} \times [\ldots] + \text{hom.}$$



JRY+ (2023)

Quadratic Quasinormal Modes

Huge progress in only a couple years!



(Courtesy of H.Zhu)

Frequency-domain in Schwarzschild

Bucciotti, Kuntz

Frequency-domain in Kerr

Khera, Ma, Yang

Accurate comparison against NR

Zhu, Pretorius

Beyond (2,2,0)x(2,2,0)

Cheung, Giesler+

Dependence on initial data (precession)

Bourg+

High-frequency limit

Bucciotti+ (JRY)

Tail effects

Cardoso+ (JRY)
Ling+

Nonlinear Perturbation Theory

What is the horizon?

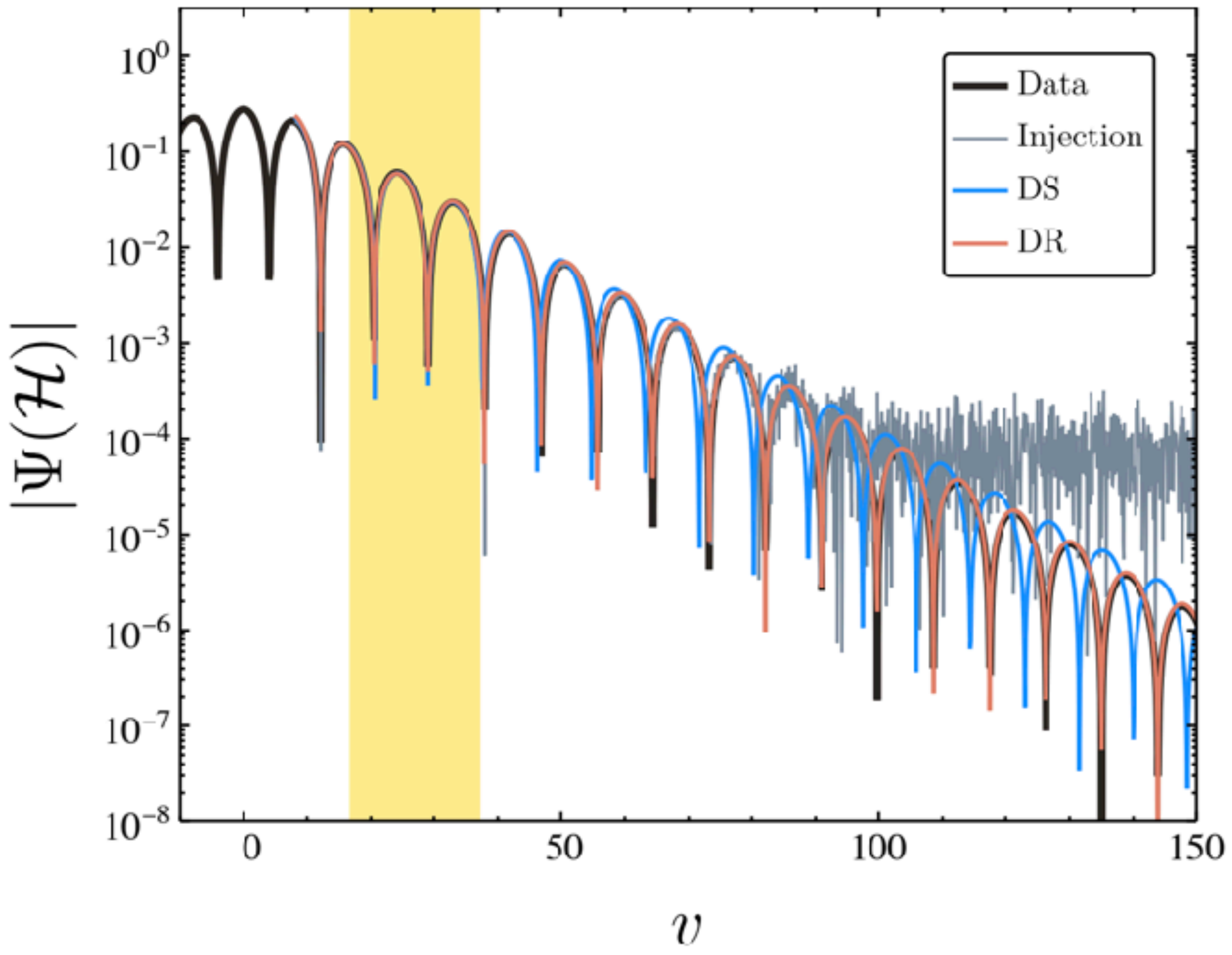
Q1 — Third order effects:

Resonances!

Coupling to non-radiative modes!

Metric reconstruction with non-compactly supported sources

Sberna+ (2021), JRY+ (2023), Zhu+ (2024), May+ (2024)



On-going w/ D.Pereñiguez & K.Fransen

Q2 — High frequency limit, approximations

Q3 — Near-horizon, near-extremal limits

Iuliano,Hollands&Green (2025)

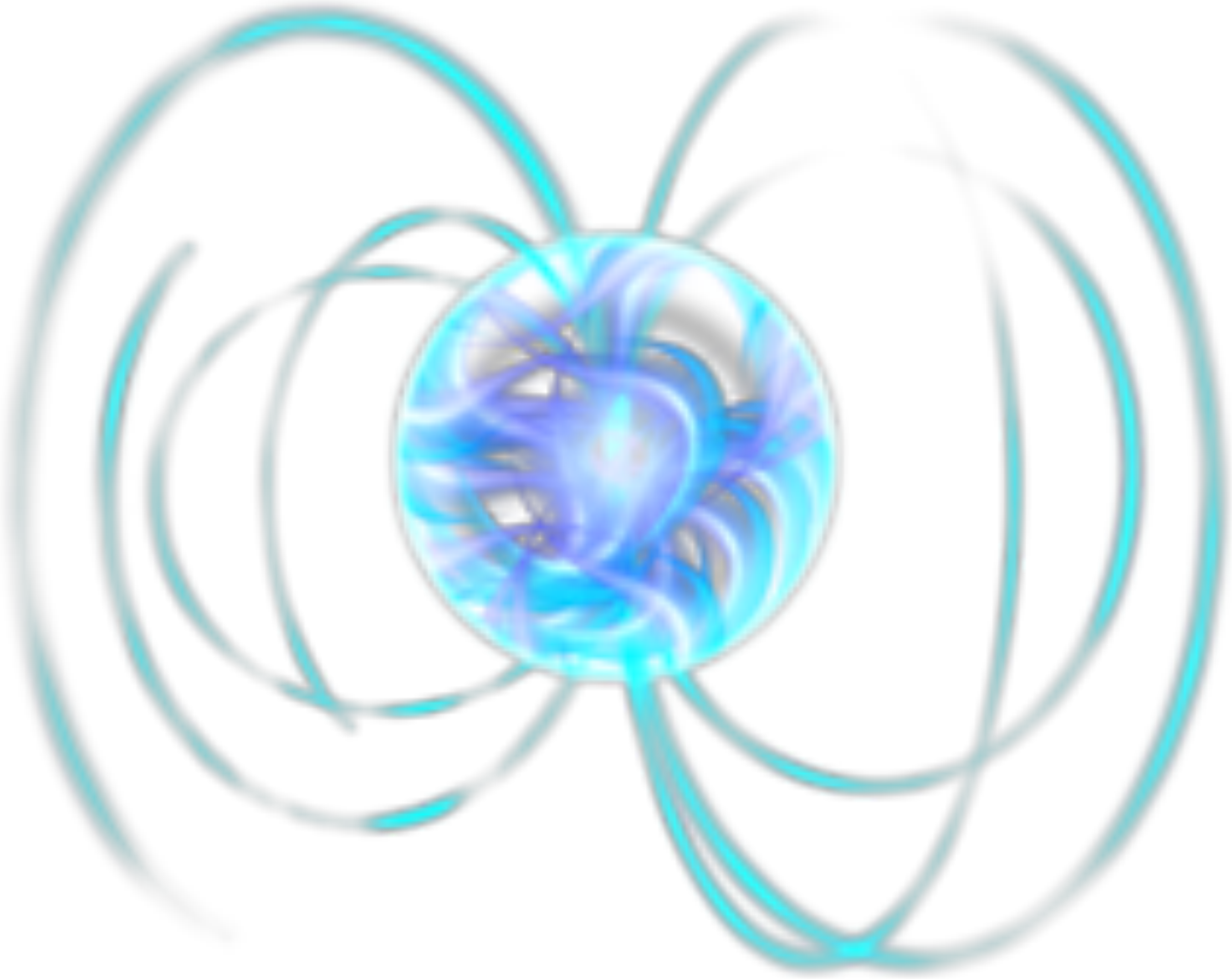
Q4 — Couplings to matter fields (Kerr-Newman?)

V. GWs and Matter

Hole of Light (2014), Levi van Veluw

Scattering waves off stars

Monochromatic, axial, non-spherical GWs impinging a compact star



$$\psi'' + (\omega^2 - V)\psi = 0$$

$$\psi \sim e^{-i\omega r} + \mathcal{R} e^{i\omega r}$$

Relativistic, dissipative hydro

Relativistic Navier-Stokes is unstable (violates causality) *Hiscock & Lindblom (1984)*

Israel-Stewart

Option 1 — “fix the equations”, by casting them in a telegraph-equation form
This makes the theory 2nd order in gradients!

Bemfica-Disconzi-Noronha-Kovtun

Option 2 — “fix the frame”, changing it until equations are stable.
This adds more transport coefficients, but keeps the theory 1st order

BDN (2021)
Kovtun (2019)

Relativistic, dissipative hydro

BDNK Hydro is described by the following stress energy tensor

$$T_{\mu\nu} = \mathcal{E} u_\mu u_\nu + \mathcal{P} \perp_{\mu\nu} + u_\mu \mathcal{Q}_\nu + \mathcal{Q}_\mu u_\nu + \mathcal{T}_{\mu\nu} ,$$

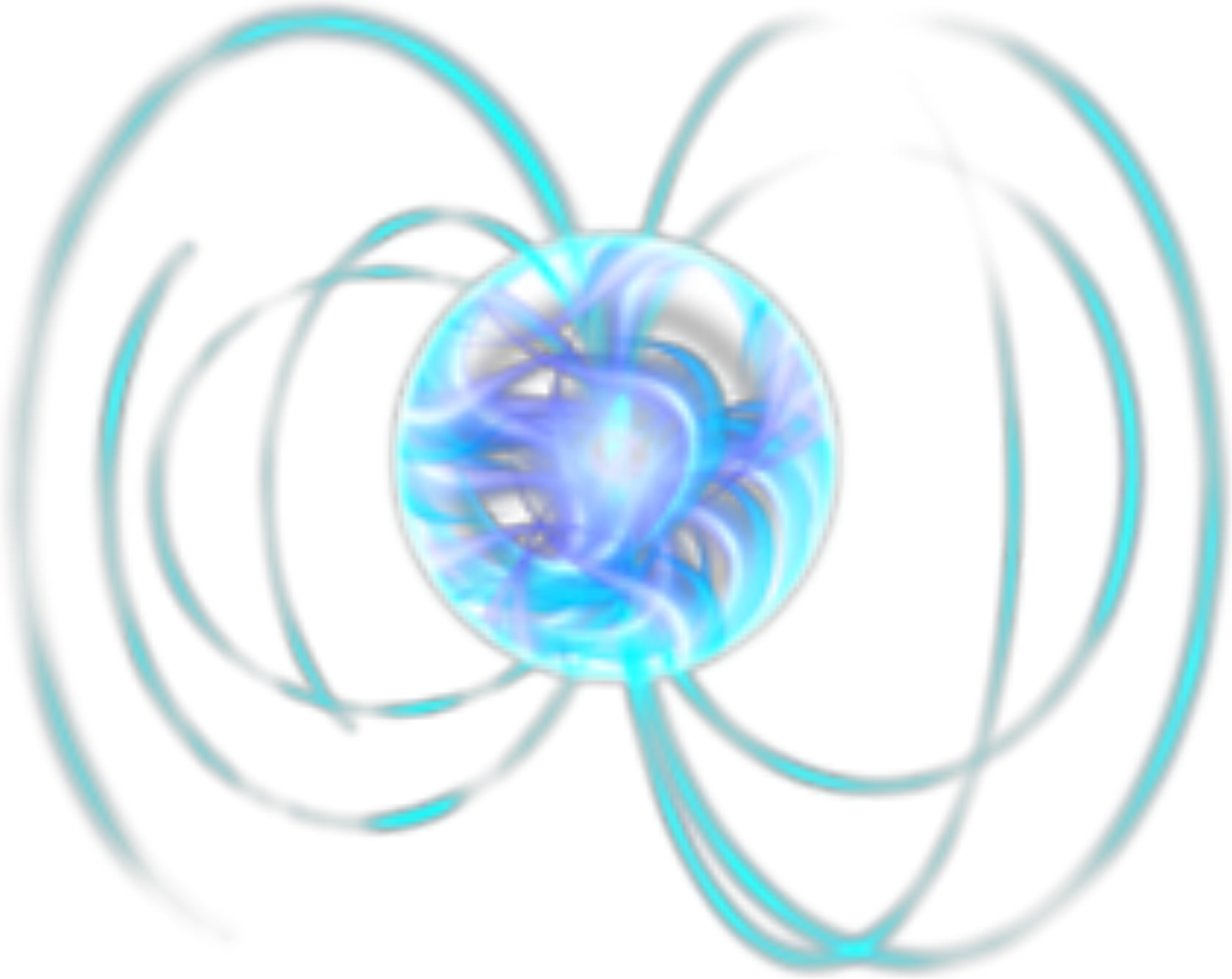
$$\mathcal{E} = e + \tau_{\mathrm{e}} \left[u^\mu \nabla_\mu e + \rho \nabla_\mu u^\mu \right] , \qquad \mathcal{P} = p - \zeta \nabla_\mu u^\mu + \tau_{\mathrm{p}} \left[u^\mu \nabla_\mu e + \rho \nabla_\mu u^\mu \right] ,$$

$$\mathcal{Q}_\mu = \tau_{\mathcal{Q}} \left[\rho u^\nu \nabla_\nu u^\mu + c_s^2 \perp_{\mu\nu} \nabla^\nu e \right] , \qquad \mathcal{T}_{\mu\nu} = -2\eta\sigma_{\mu\nu} ,$$

5 (+1) transport coefficients. Causality implies certain constraints among them.

Scattering waves off viscous stars

Axial fluid perturbations become dynamical — coupling between gravity and viscous modes

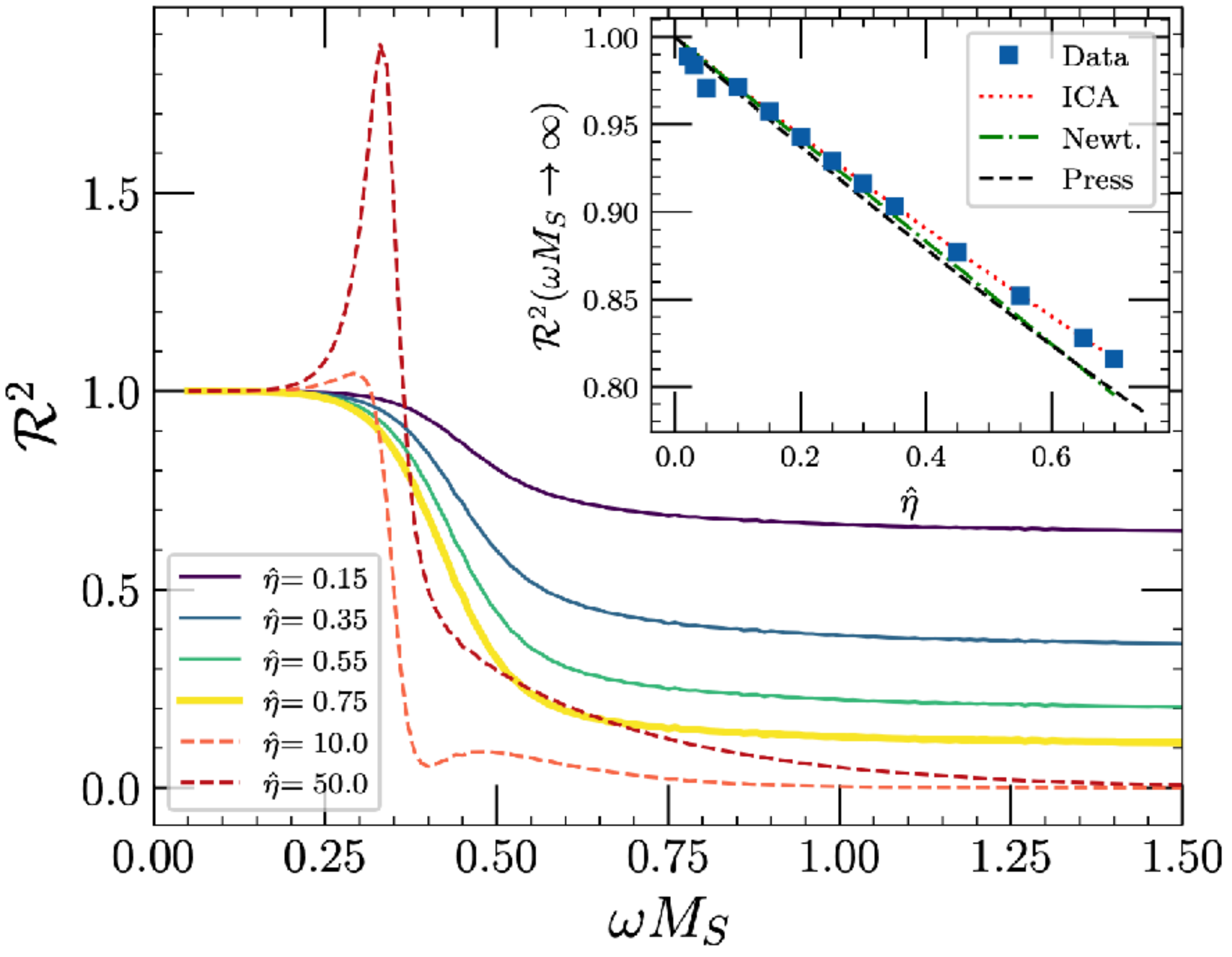


$$\psi'' + (\omega^2 - V)\psi = \eta(i\omega\psi + A\varphi)\,,$$

$$\varphi'' + (c_\eta^{-2}\omega^2 - V_\eta)\varphi = \ldots\,,$$

JRY (2024)

Scattering waves off viscous stars



Viscosity absorbs waves.

(Acausal) viscosity can reflect GWs too!

High Freq limit recovers Press'!

$$k^2 = \frac{\omega^2}{c^2}\left(1 + i\,\frac{16\pi G\eta}{\omega}\right)$$

Rotation induces superradiance
(stay tuned!)

Scattering waves off viscous stars

On Gravitational Conductors, Waveguides, and Circuits[1,2]

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Received November 6, 1978

Abstract

We show that a material with sufficiently large elastic shear modulus or shear viscosity will act like a gravitational conductor or "metal." It will reflect gravitational waves, and it can be used to make gravitational waveguides and circuits. Unlike electromagnetism, a gravitational wave can be guided by a single conductor in transverse mode. Gravitational conductors can obey the dominant energy condition, and they can be larger than their Schwarzschild radius, but they must violate a new condition that is probably satisfied by all existing forms of matter. Direct-current gravitational circuits, although limits of guided gravitational waves, have a simple Newtonian interpretation.

Viscosity interacts in a non-trivial way with GWs. Astrophysical scenarios?

VI. Conclusions

Hole of Light II (2014), Levi van Veluw

Take-aways

- Gravitons / Gravitational waves do not propagate freely

- Self-interactions can be important:
  - At resonances (4-wave interactions, e.g. AdS)
  - Near strong fields (quadratic QNMs)

- Systematically include these effects in waveforms
  - Will always need solution to 2-body problem?
  - White-hole fission picture may help?

- GWs couple to (hydro) matter through viscosity
  - Astrophysical or cosmological regimes?

Thank you for your attention!

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