Nonlinear effects in perturbation theory

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Danmarks Grundforskningsfond
Danish National Research Foundation

Outline

1. Beyond linearised gravity
2. An illustrative example
3. Spinors and black hole perturbations
4. Nonlinearities in plane waves
5. Take-aways

I. Beyond linearised gravity

3-Iron (빈집), 2004, Kim Ki Duk (김기덕)

Beyond linearised gravity

Let us assume a split between a foreground and a background

$$g_{ab} = \bar{g}_{ab} + h_{ab} + k_{ab}$$

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In some coordinates (***steady coordinates, MTW***), we have

$$|\partial_a \bar{g}_{bc}|^2 \sim |\bar{R}_{abcd}| \sim |g_{bc}|/\mathscr{R}^2$$

$$|h|_{ab} \sim \mathscr{A}\,|g|_{ab}$$

$$|\partial_a h_{bc}| \sim |h_{bc}|/\lambda$$

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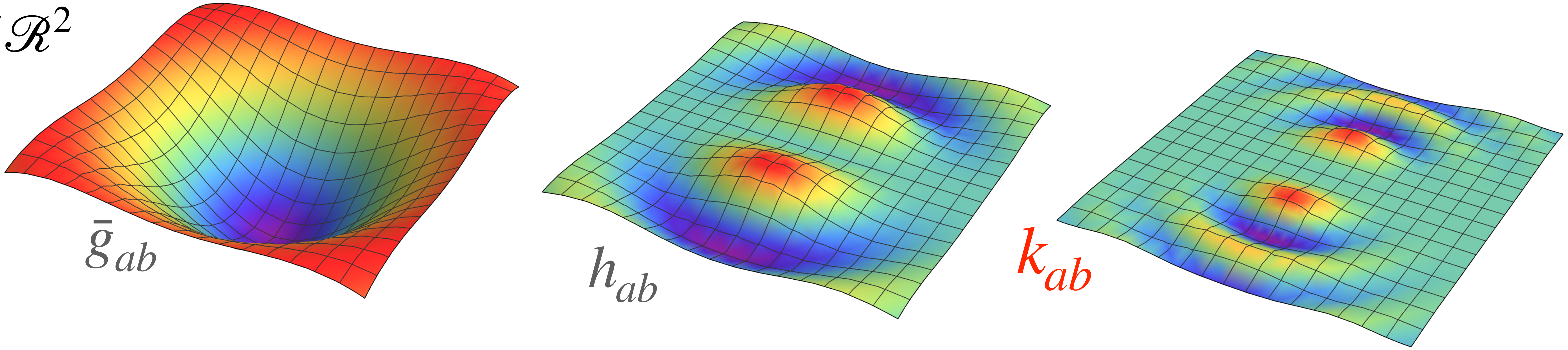

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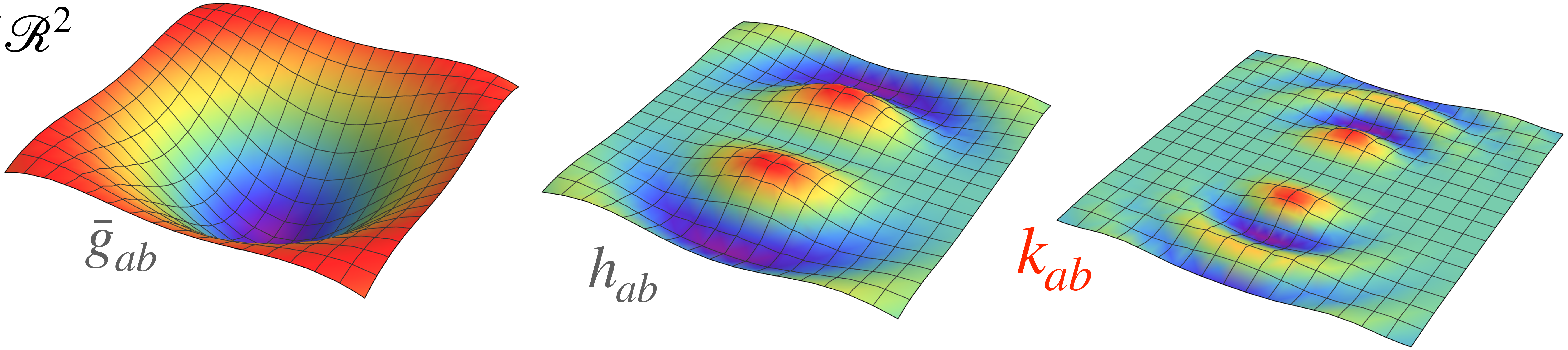

Perturbation theory holds if $\mathscr{A} \lesssim \lambda/\mathscr{R}$

Beyond linearised gravity

Plugging this into Einstein Equations, in the ***harmonic gauge**** leads to

$$\Box h_{ab} = 0$$

$$\Box k_{ab} = 2\partial_{(a}h^{cd}\partial_c h_{b)d}$$

Isaacson (1967)

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If the linearised metric describes a plane wave

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$$h_{ab} \sim e^{i\omega(t-z)} \epsilon_{ab}$$

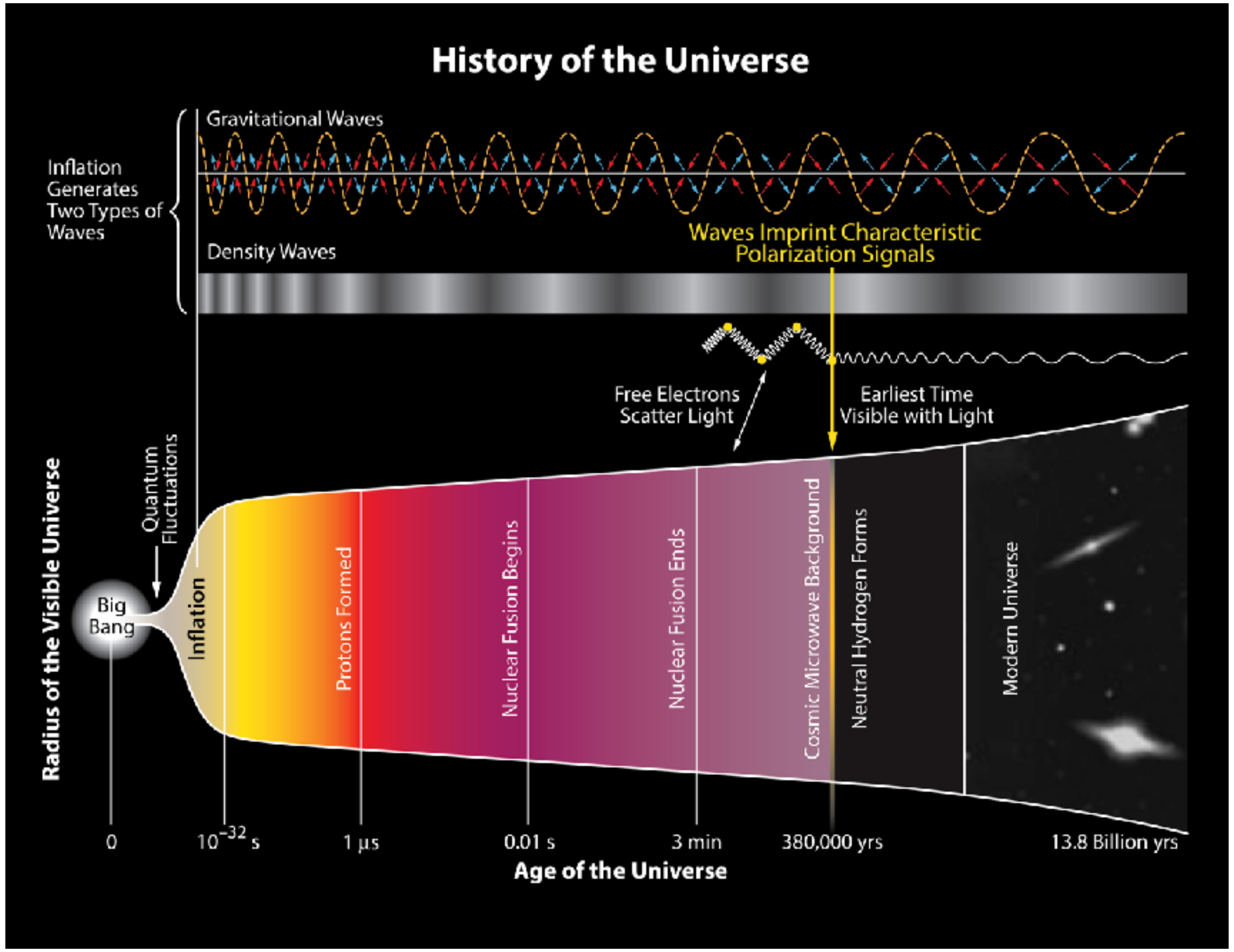
It excites ***higher harmonics*** as it propagates

$$k_{ab} \sim e^{2i\omega(t-z)} \langle \epsilon_{ab} \,|\, \partial_{(a} \epsilon^{cd} \partial_c \epsilon_{b)d} \rangle + \ldots$$

Isaacson (1967)

Nonlinearities abound

Early Universe — ***Scalar-induced GWs*** important to discern inflationary models

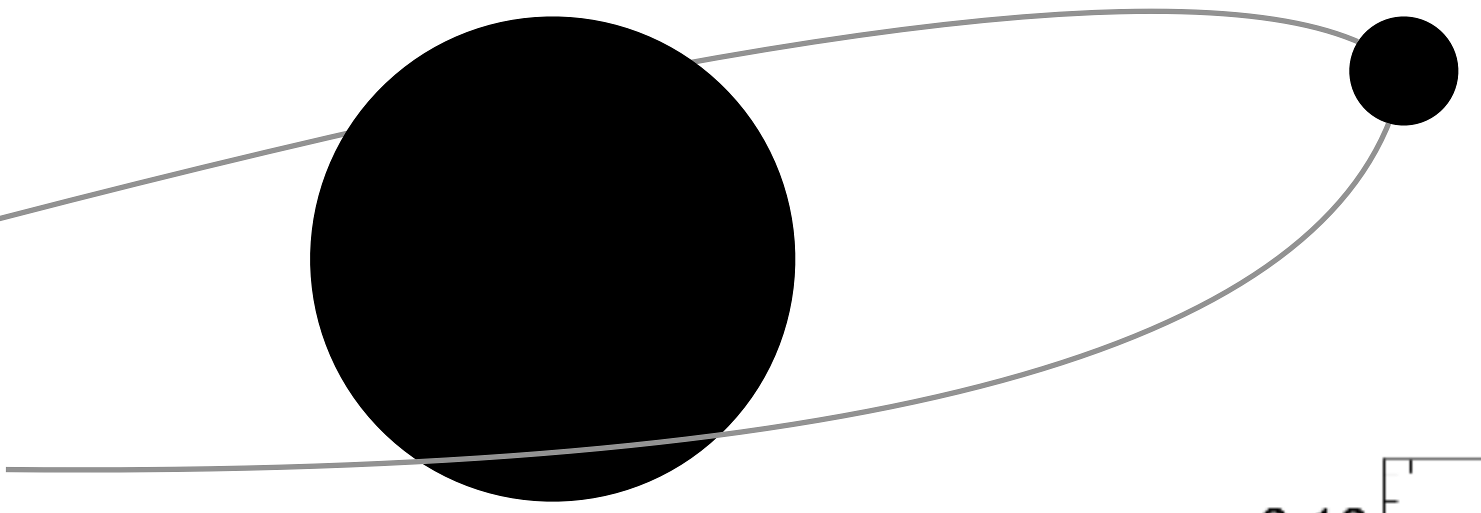


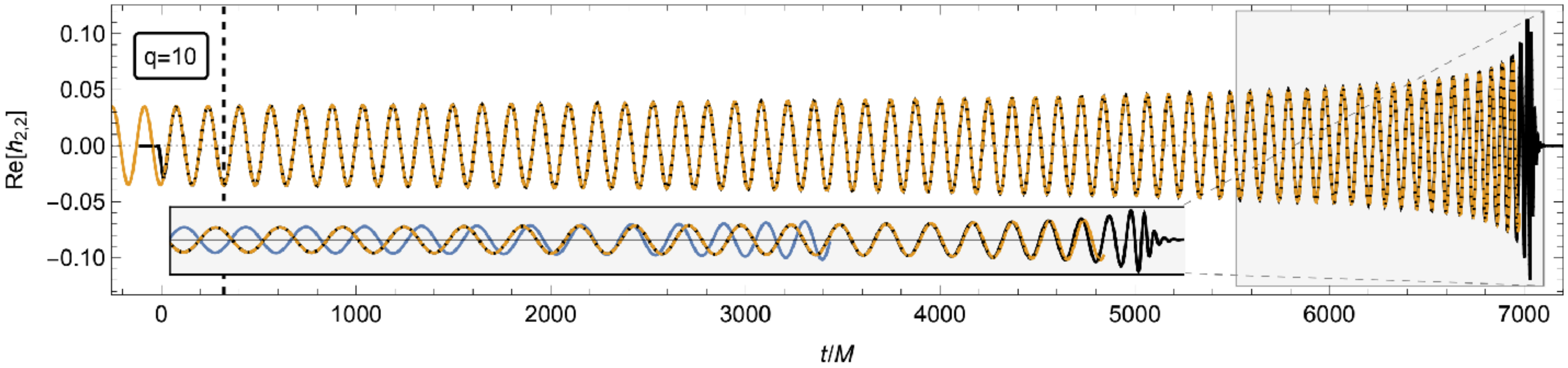
BICEP-2

Tomita (1967)
Baumann+ (2007)

Nonlinearities abound

Extreme Mass Ratio Inspirals — In band for $\mathcal{O}(\nu^{-1})$ cycles, so $\mathcal{O}(\nu^{2})$ accuracy is needed





Wardell+ (2023)

Nonlinearities abound

Instability of anti-de Sitter — generic ***black hole formation*** from small data

Nonlinearities abound

Stability of trapping spacetimes — black hole mimickers, Kerr-AdS, black rings…

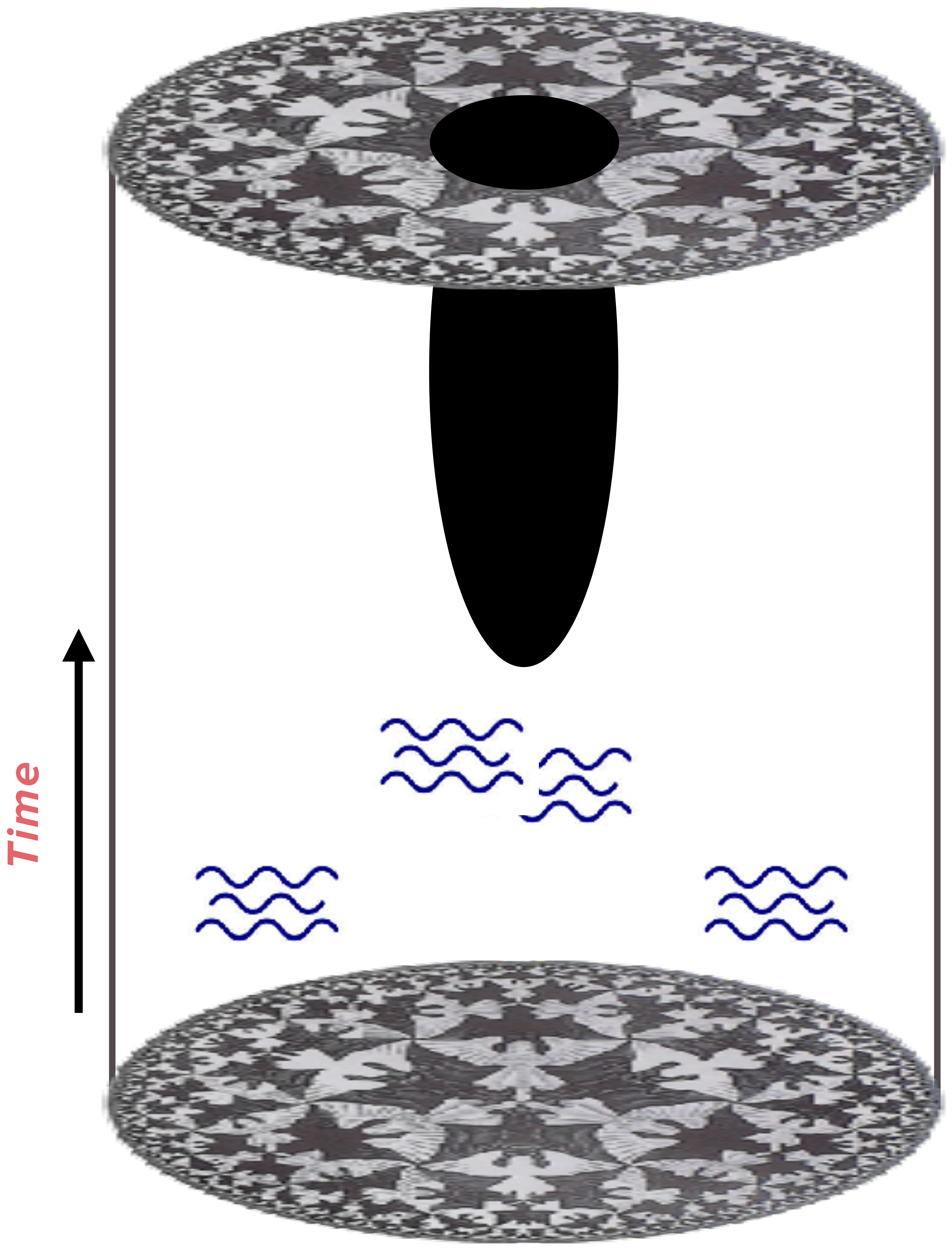
Nonlinearities abound

Black Hole Ringdown — ***quadratic quasinormal modes***, memory, etc

II. An illustrative example

Spring, Summer, Fall, Winter… and Spring (봄 여름 가을 겨울 그리고 봄)
Kim Ki Duk (김기덕)

Instability of anti-de-Sitter



Pure AdS is generically* ***unstable*** to scalar fluctuations

$$\Box \Phi^{(1)} = 0$$

$$\partial_x^2 \Phi^{(1)} + (\omega^2 - V)\Phi^{(1)} = 0$$

Bizon+ (2011-)
Dias+ (2012+)
Buchel+ (2013)

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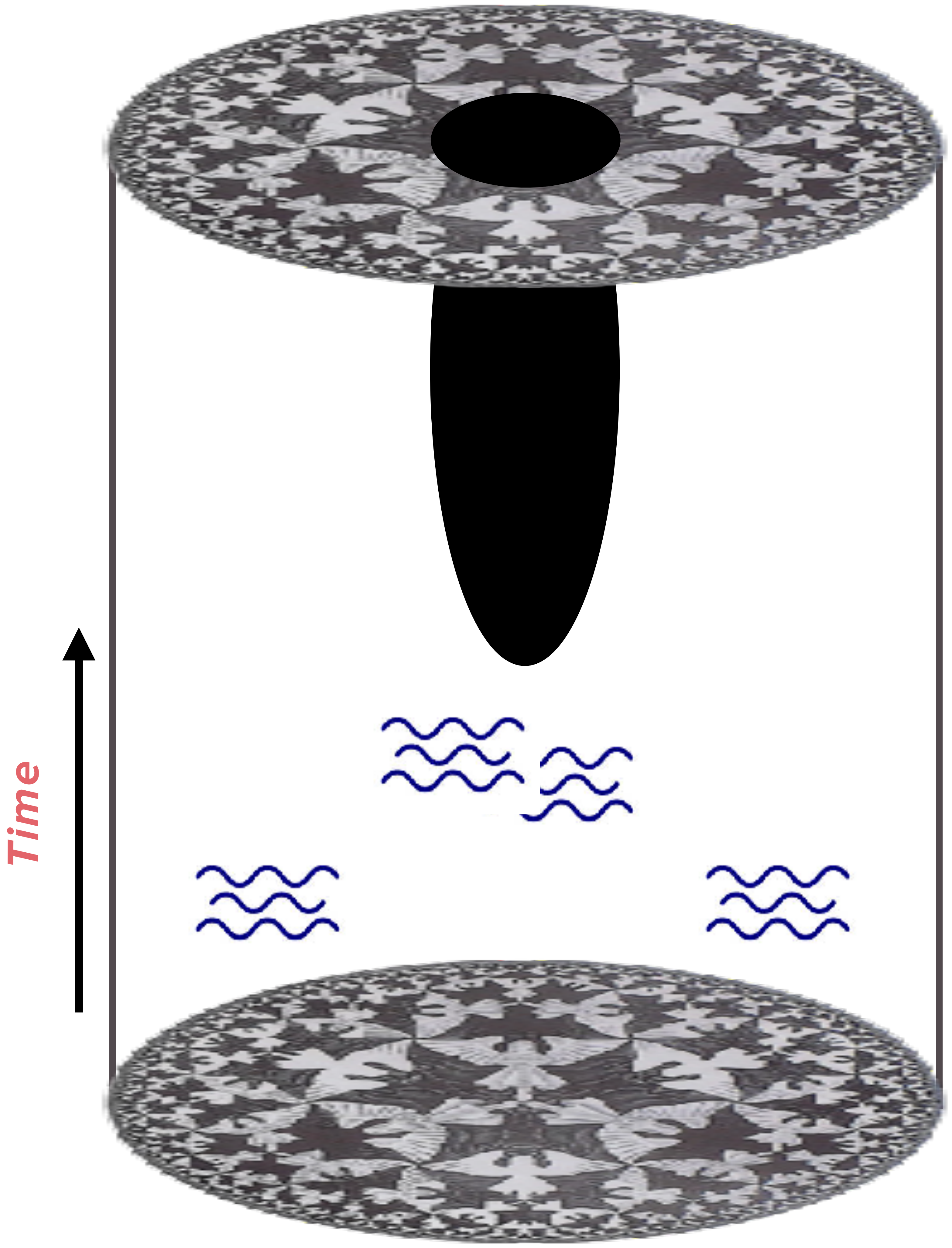

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With reflective (Brown-Henneaux) boundary conditions

$$\Phi^{(1)} = \sum_{n=0}^{\infty} A_n e^{-i\omega_n t} e_n(x)$$

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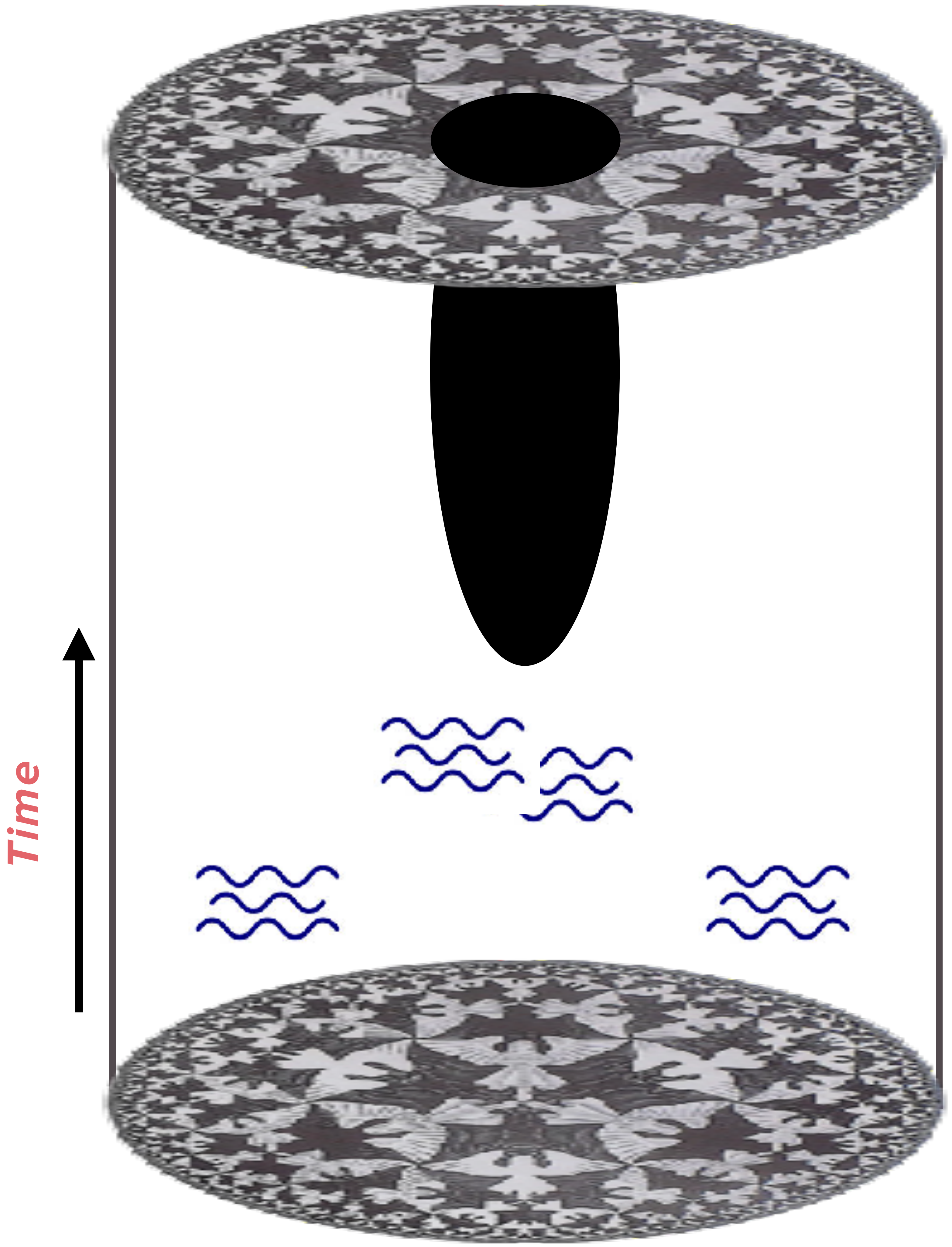

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$$\Phi^{(1)} = \sum_{n=0}^{\infty} A_n e^{-i\omega_n t} e_n(x)$$

$$\omega_\ell^2 = (3n+1)^2$$

The sum of 2 frequencies is also in the spectrum!!

Bizon+ (2011-)
Dias+ (2012+)
Buchel+ (2013)

Instability of anti-de-Sitter

Back-reaction on the metric is not dynamical

$$ds^2 = \sec^2 x\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x d\Omega^2\right)$$

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To third order…

$$\partial_x^2 \Phi^{(3)} + (\omega^2 - V)\Phi^{(3)} = \mathcal{S}(\Phi^{(1)}, A^{(2)}, \delta^{(2)}) \sim e^{-i\omega_n t}$$

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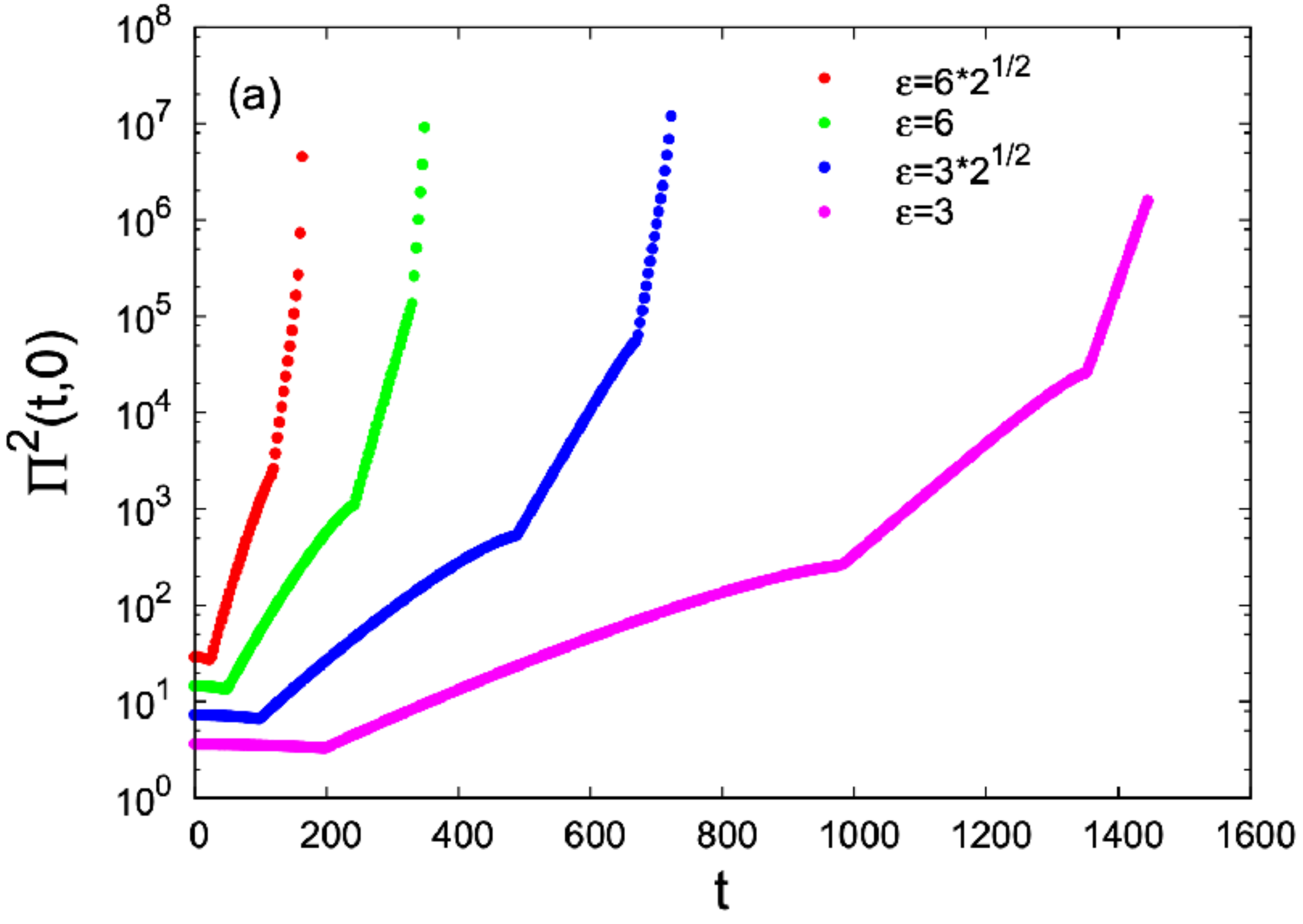
Resonance —> secular growth $\quad \Phi^{(3)} \sim te^{-i\omega_n t} + \ldots$



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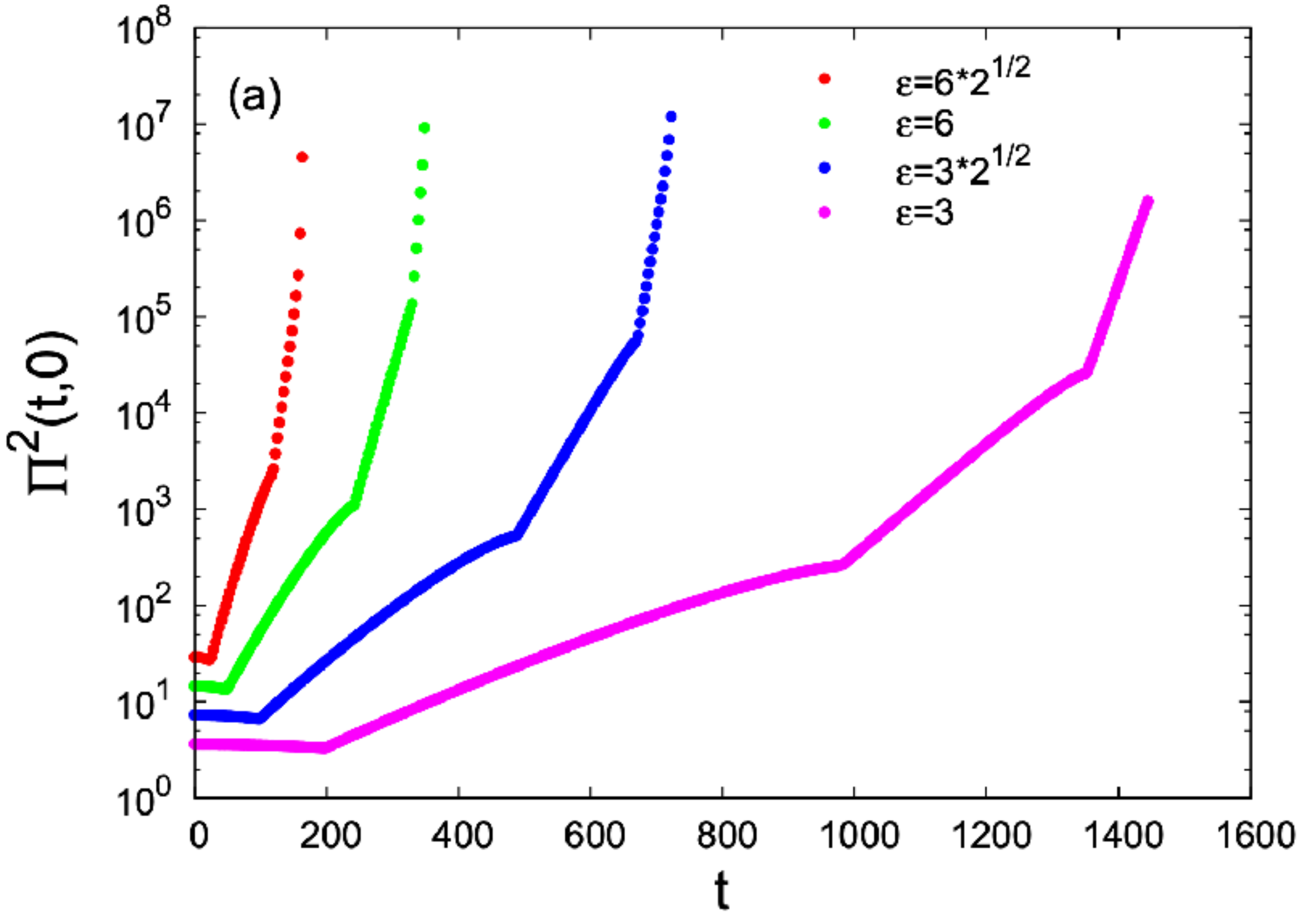
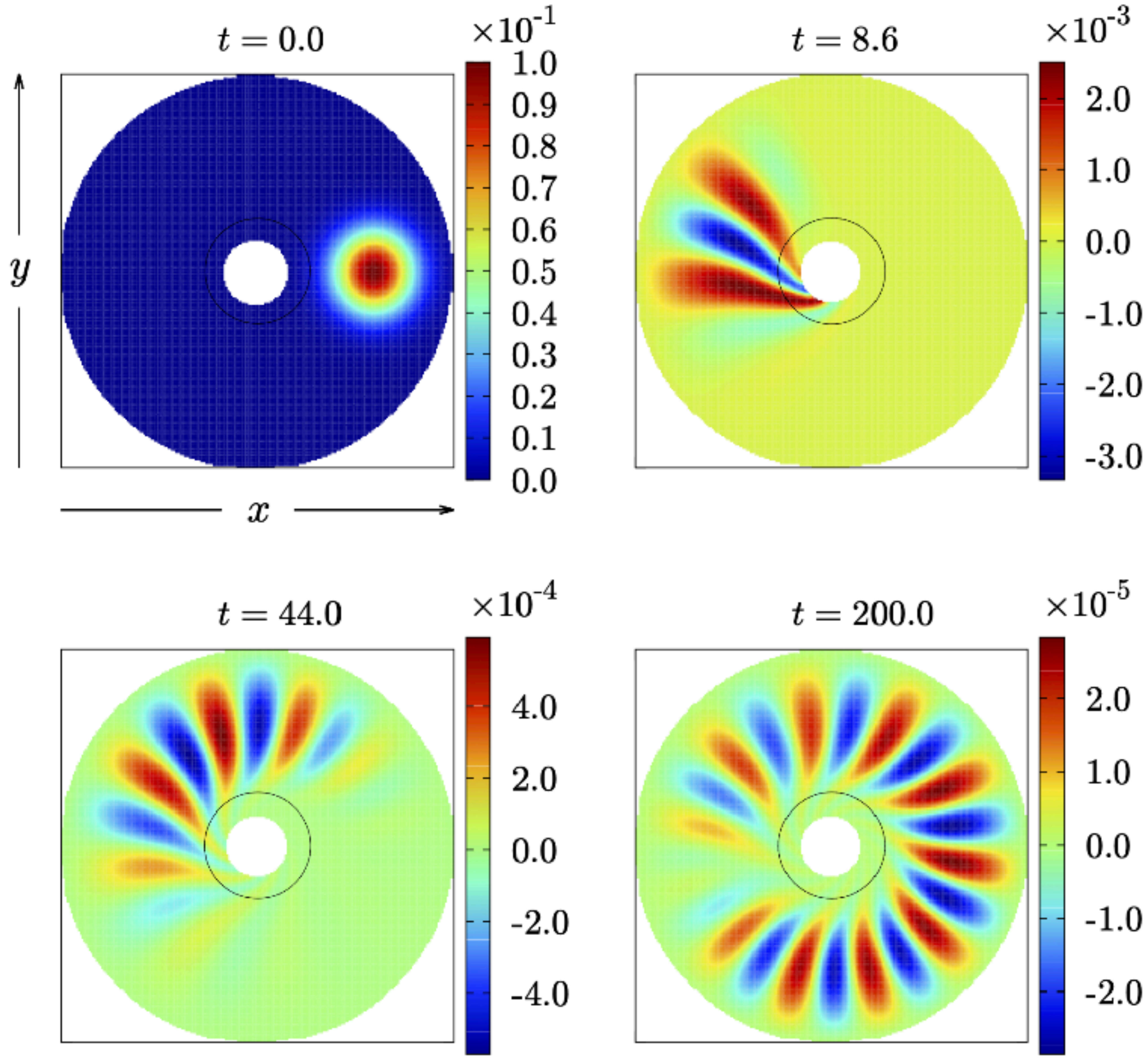


AdS unstable towards Kerr AdS

Bizon+ (2011-)
Dias+ (2012+)
Buchel+ (2013)

Trapped waves

Kerr-AdS traps radiation between the centrifugal barrier & AdS boundary — is it stable?



Figueras & Rossi (2023)

Keir (2013)
Cardoso+ (2014+)
Cunha+ (2019)
JRY+ (2025)
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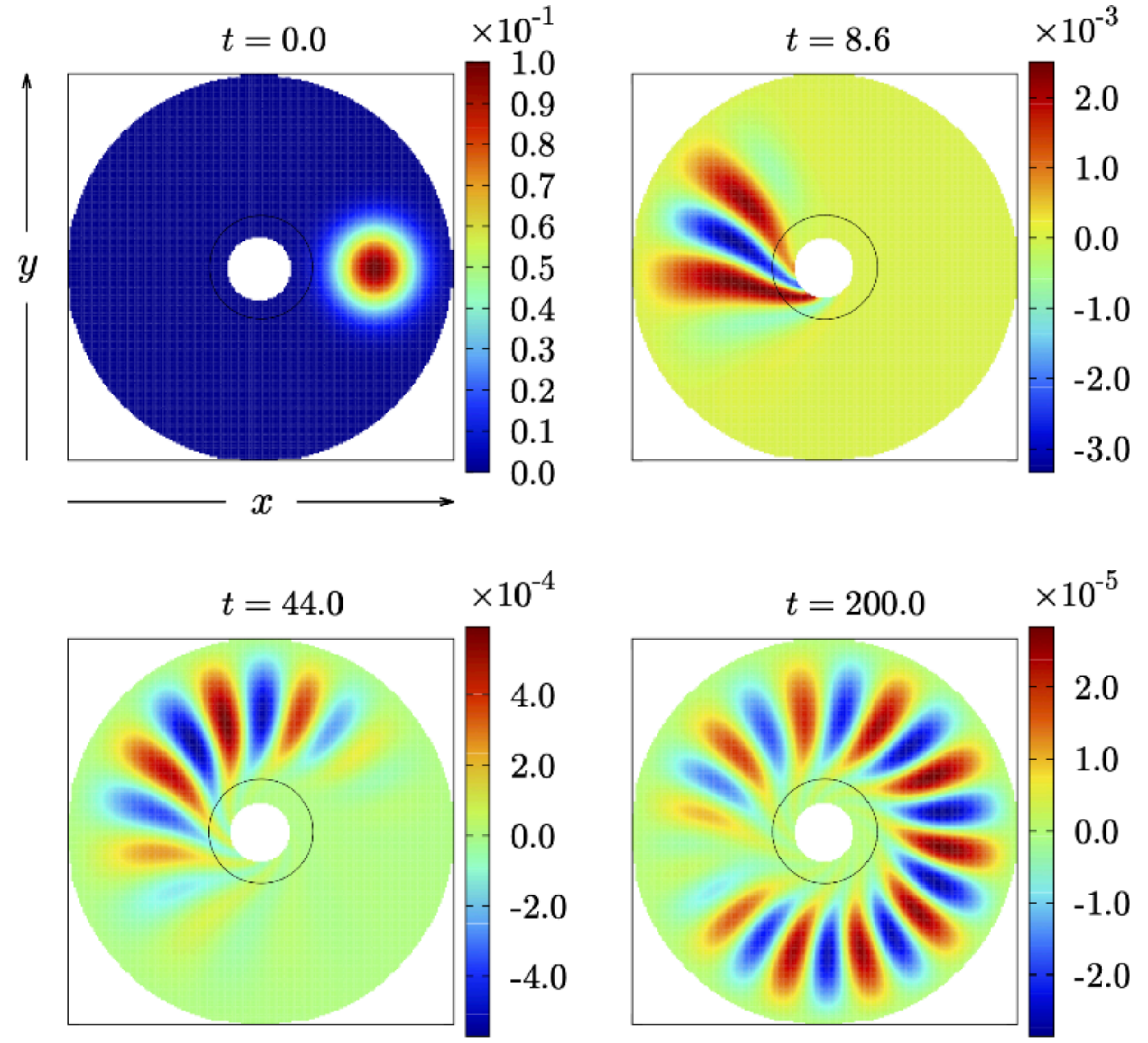
Similar behaviour in BH mimickers, or black compact objects in higher dimensions

Trapped, long-lived modes were conjectured to lead to a ***non-linear instability***



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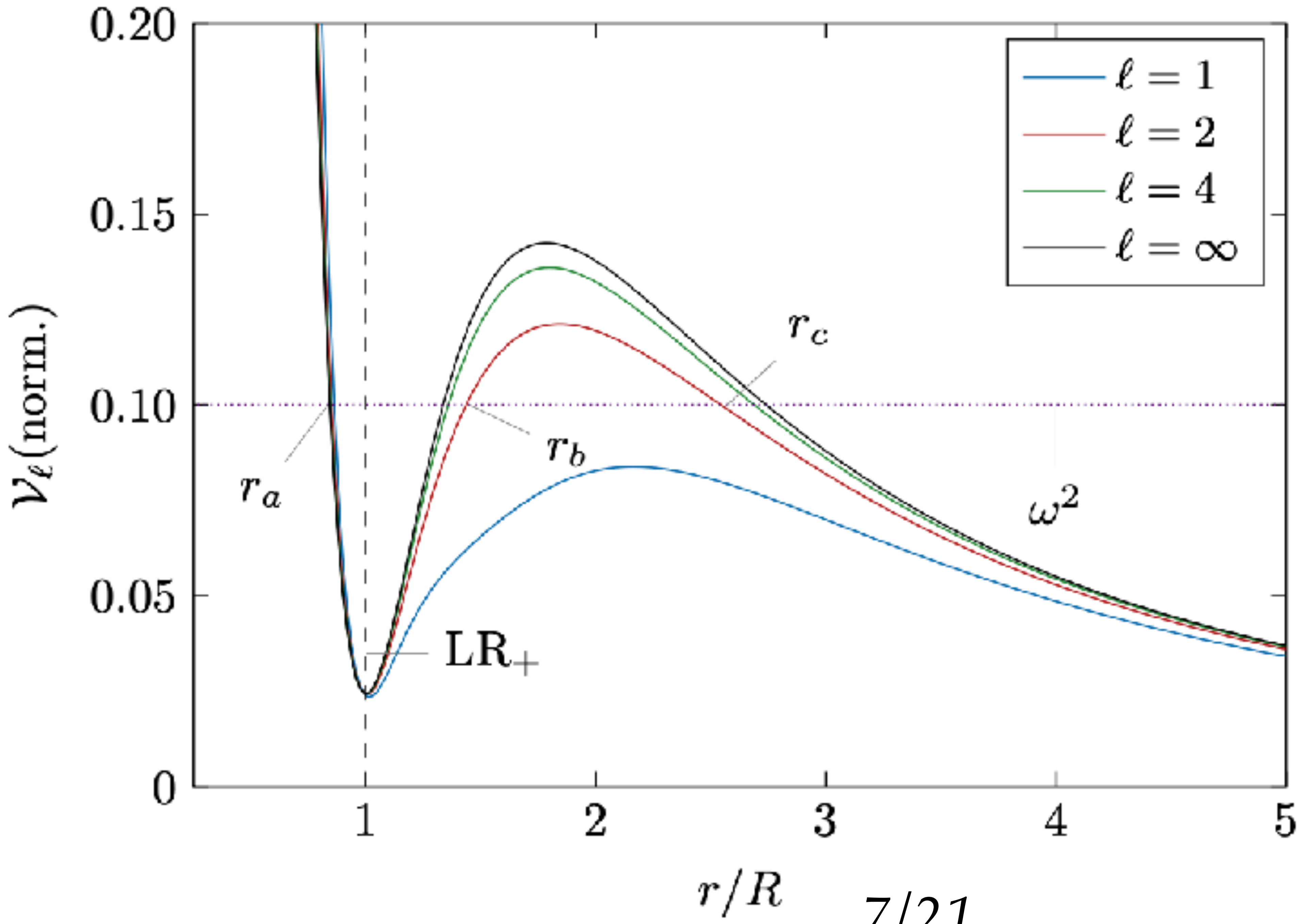
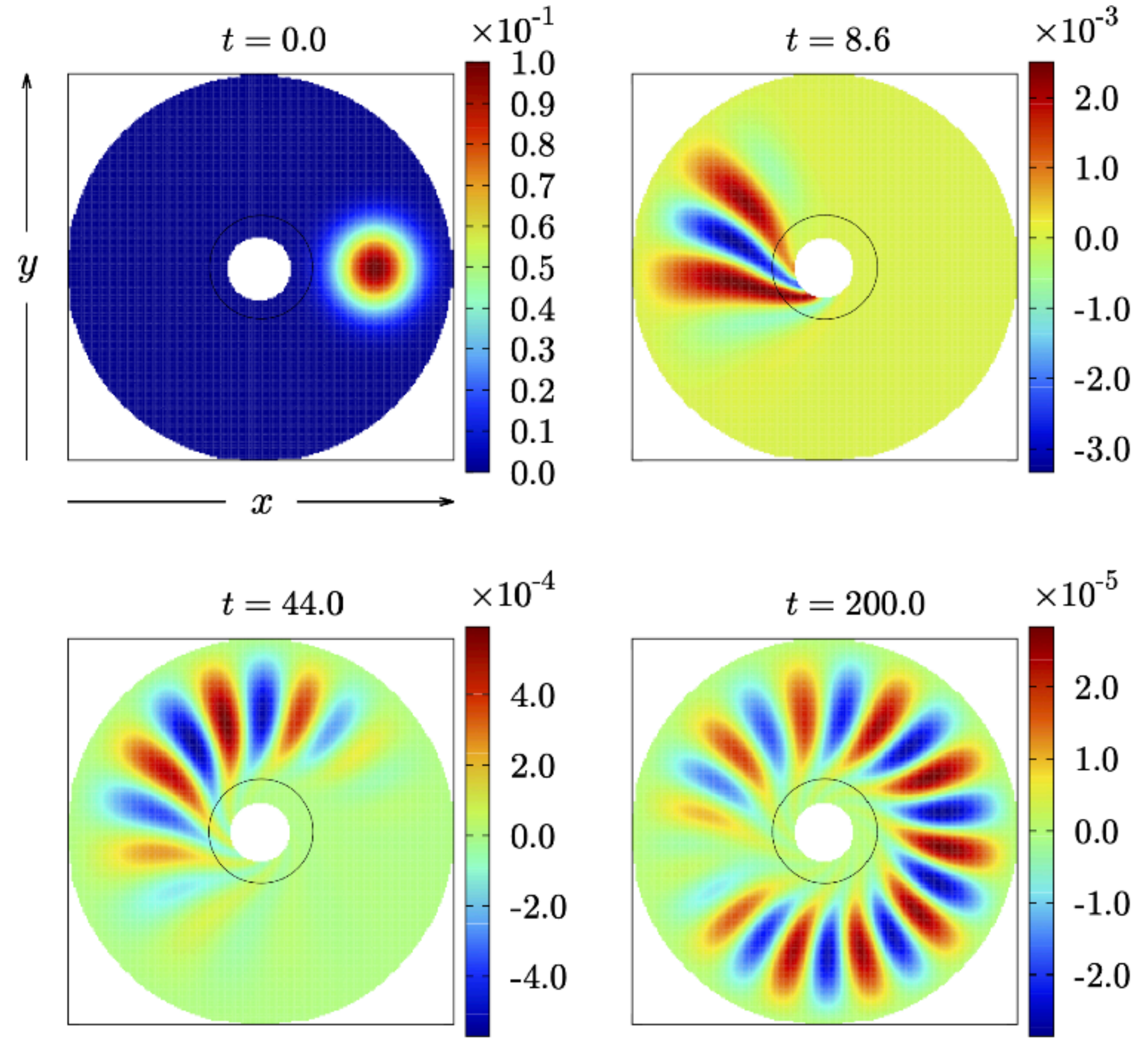
$$\Box_g \Phi = \Phi^3 \qquad \Phi_\ell'' + (\omega^2 - V_\ell)\Phi_\ell = \sum c_\ell^{123}\Phi_1\Phi_2\Phi_3$$





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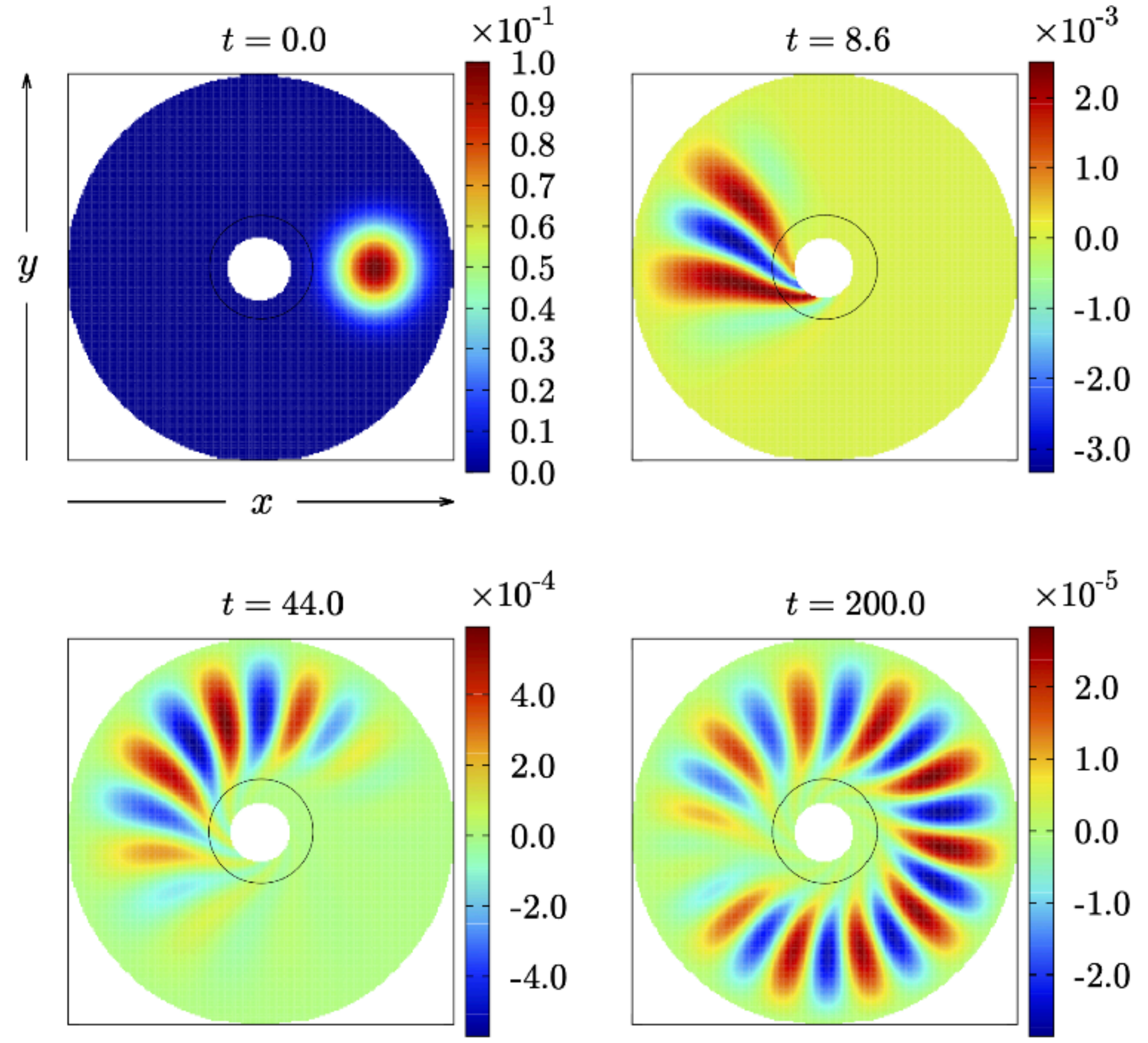
Instability? Unlikely



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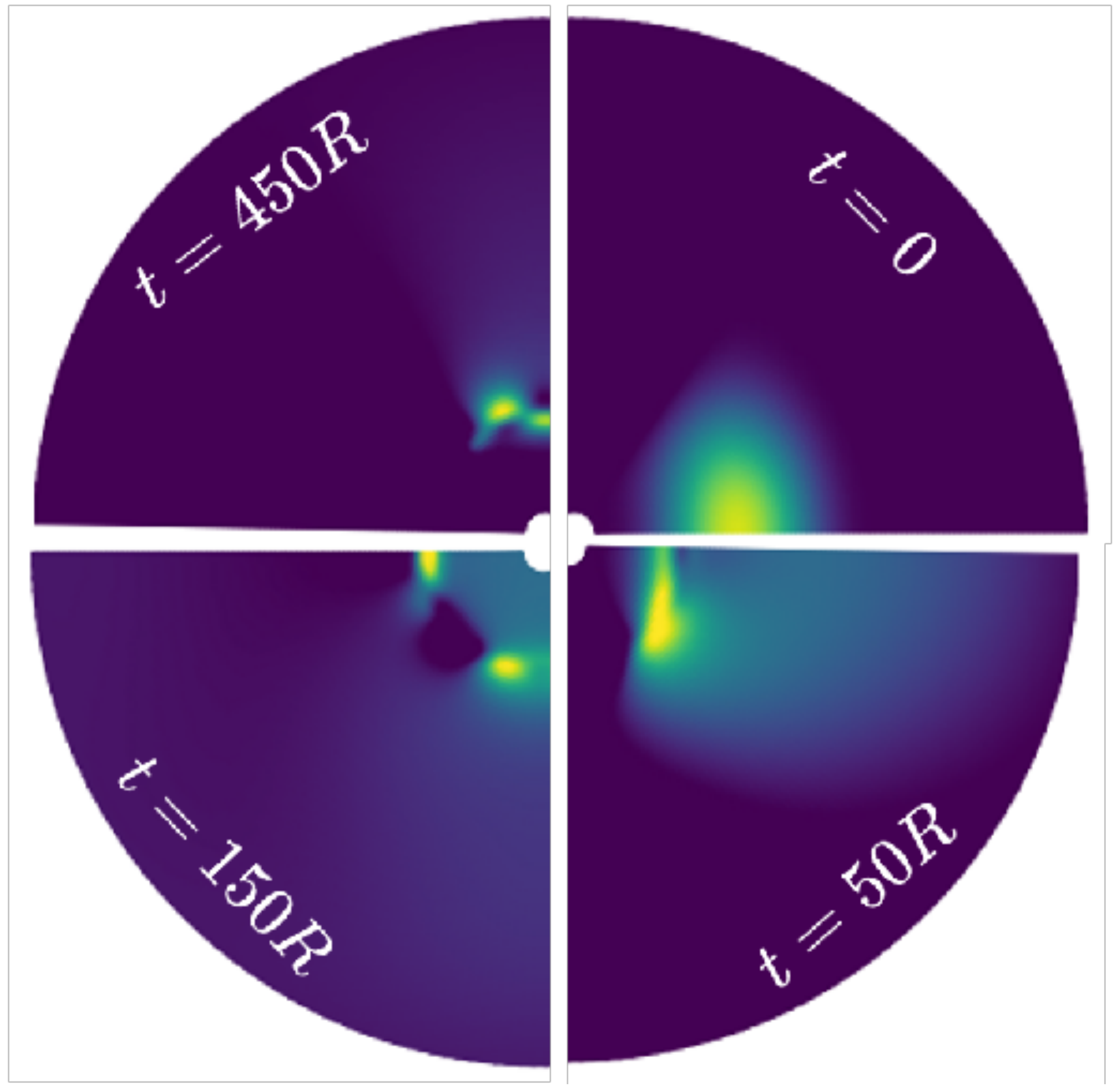

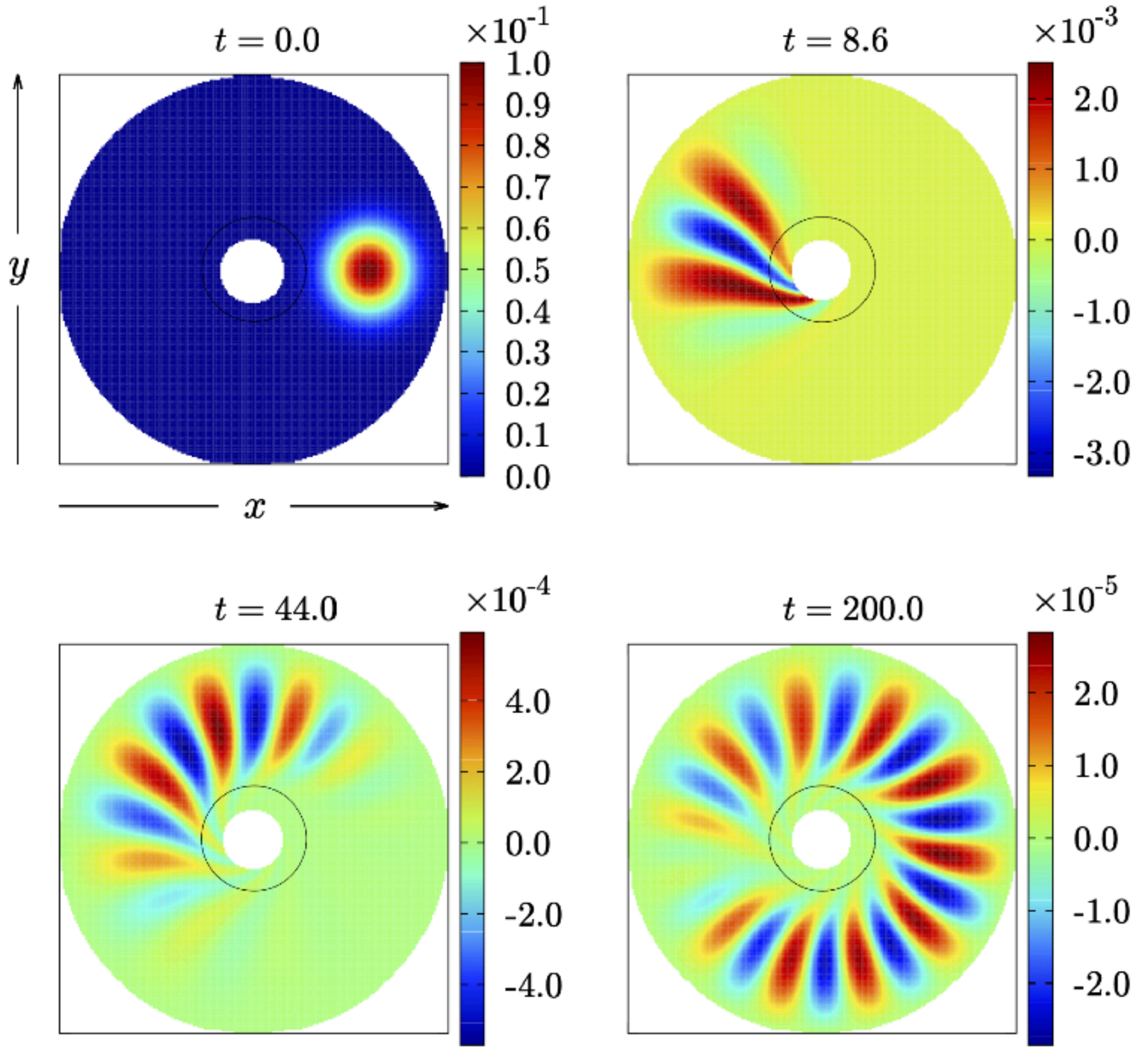

Keir (2013)
Cardoso+ (2014+)
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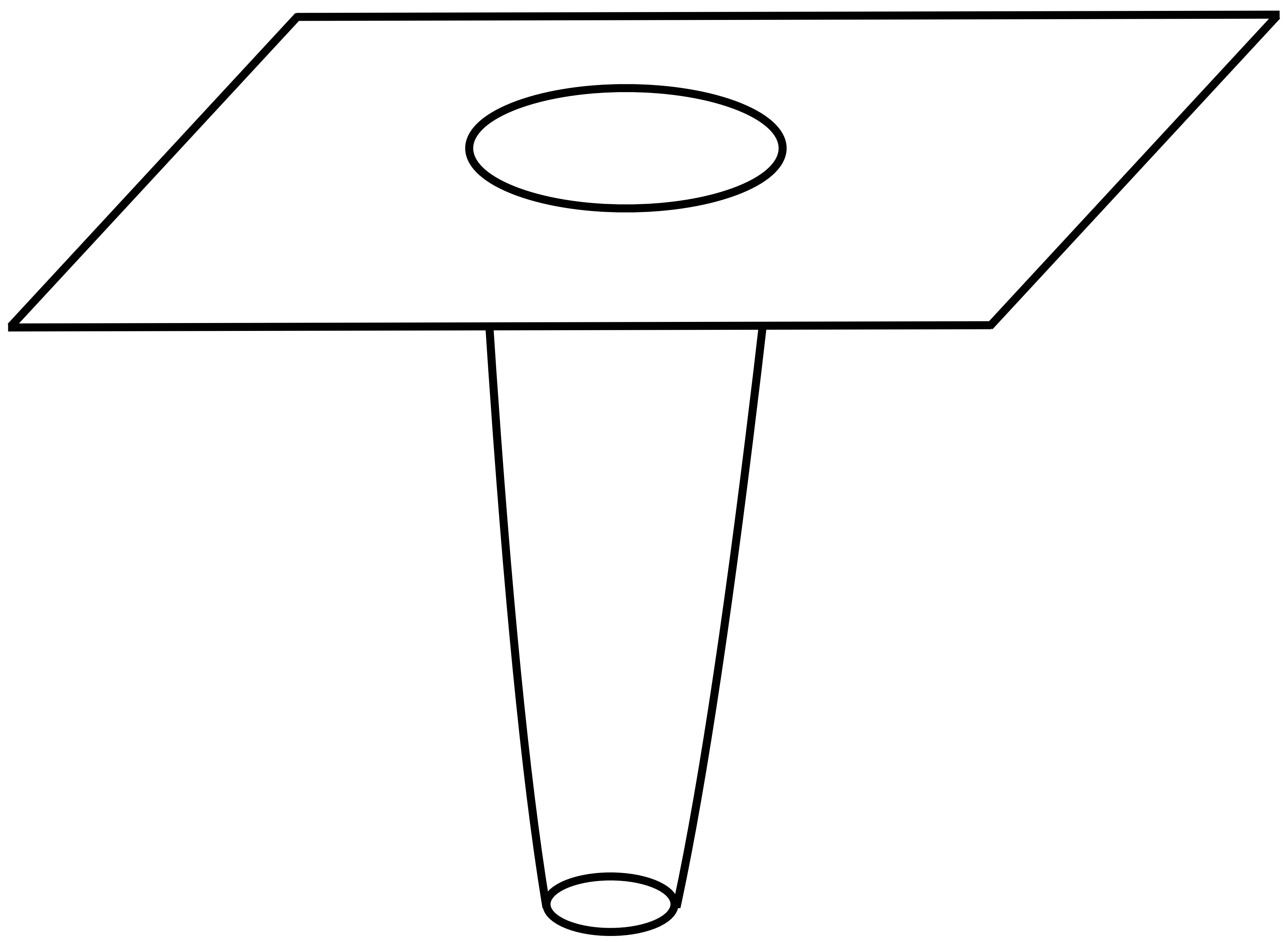
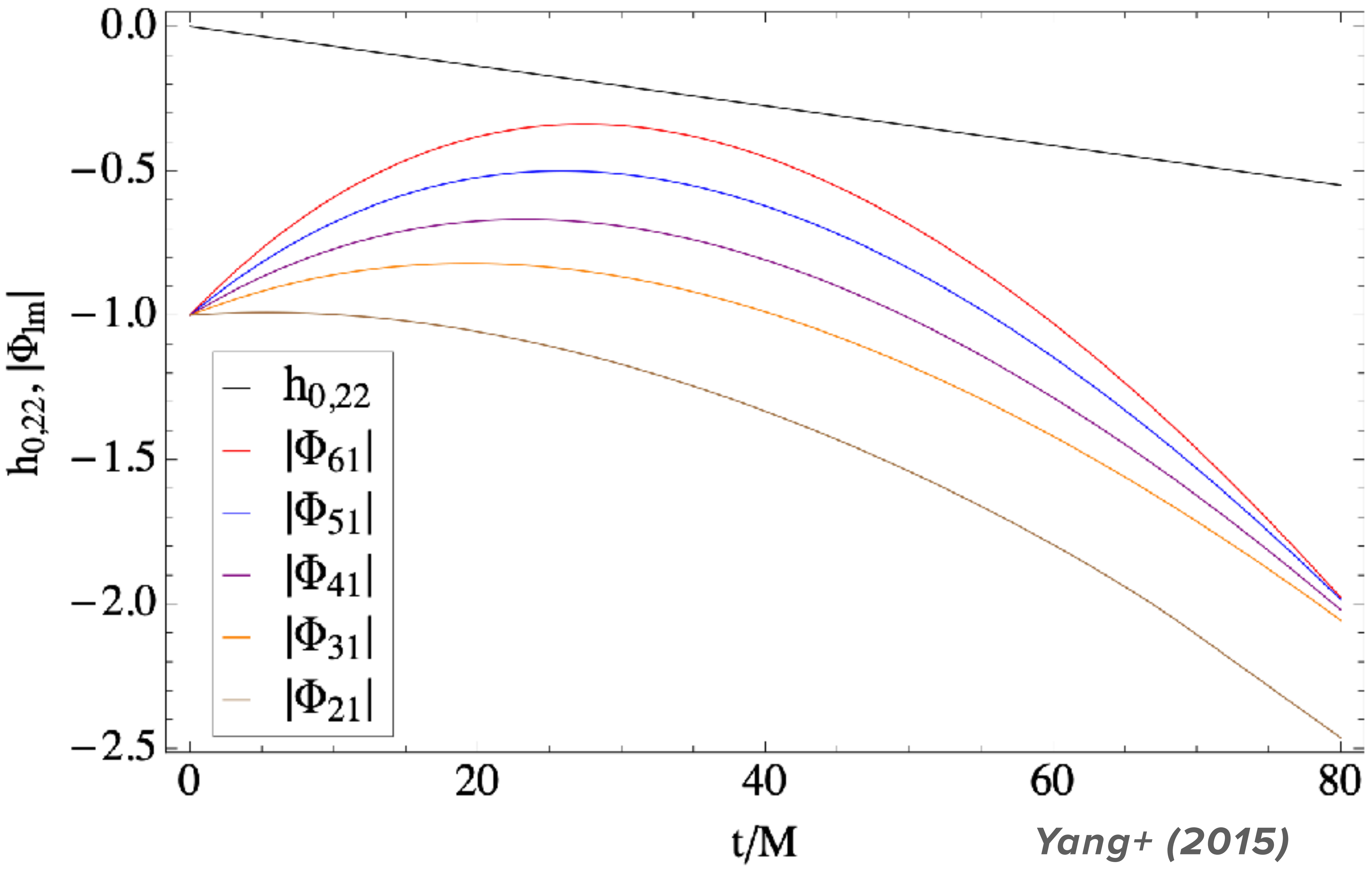
Near-extremal Black Holes

Near-extremal black holes also have long-lived modes

Hints towards turbulent behaviour in this regime—***third order*** couplings important





Yang+ (2015)

Yang+ (2015)
Iuliano (2024)

III. Spinors and BH Perturbations

Parasite (기생충) 2023, Bong Joon Ho (봉준호)

Spinor Formulation of GR

Lorentzian manifolds in 4 dimensions admit a spin structure

Isomorphism $$\Lambda(1,3) \simeq SL(2,\mathbb{C})/\{\mathbf{1} \sim -\mathbf{1}\}$$

Penrose (1960)

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NP tetrad $$\{l^a, n^a, m^a\} = \{o^A o^{A'}, \iota^A \iota^{A'}, o^A \iota^{A'}\}$$

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Curvature $$R = 24\Lambda \qquad S_{ab} = R_{ab} - \frac{1}{4}R g_{ab} = -2\Phi_{ABA'B'} \qquad C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \mathrm{c.c.}$$

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Algebraic classification

$$\Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(2)}_{B}\kappa^{(3)}_{C}\kappa^{(4)}_{D)}$$

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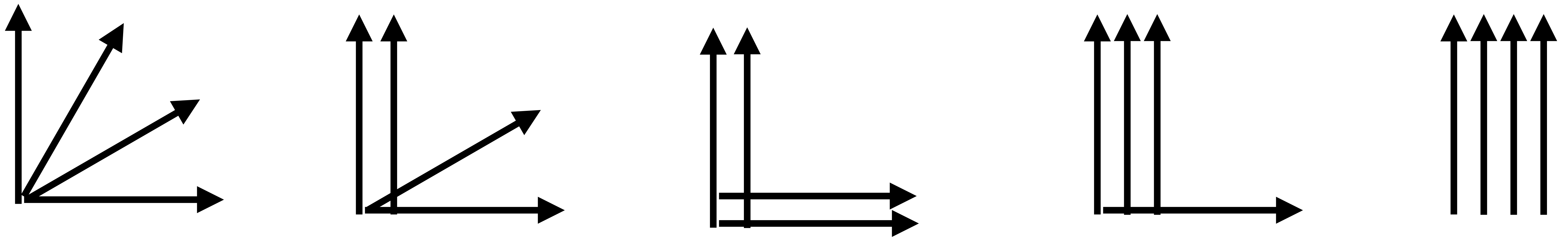

Type I ***Type II*** ***Type D*** ***Type III*** ***Type N***

Penrose (1960)

Teukolsky Equation in a Nutshell

Bianchi identities in spinor form are just $\nabla^{AA'}\Psi_{ABCD} = 0.$

Penrose (1960)
Stewart & Walker (1974)
Bini+ (2002)

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Taking one more derivative, and commuting…. $\Box\Psi_{ABCD} = 6\Psi^{EF}_{(AB}\Psi_{CD)EF}$

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Projecting with the spin dyad $\iota^A\iota^B\iota^C\iota^D \times \ldots$ we find

$$\mathcal{O}_4\Psi_4 + \mathcal{O}_3\Psi_3 + \mathcal{O}_2\Psi_2 = 0$$

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Where

$$\mathcal{O}_4 = [\text{Þ}'\text{Þ} - \eth'\eth - (4\rho' + \bar{\rho}')\,\text{Þ} - \rho\text{Þ}' + (4\tau' + \bar{\tau})\,\eth + \tau\eth' + 4\rho\rho' - 4\tau\tau' - 2\psi_2]$$

$$\mathcal{O}_3 = [4\text{Þ}\kappa' - 4\eth\sigma' - 4(\bar{\rho} - 2\rho)\,\kappa' + 4(\bar{\tau} - 2\tau)\,\sigma' + 10\psi_3]$$

$$\mathcal{O}_4 = [-4\sigma'\text{Þ}' + 4\kappa'\eth' - 12\kappa'\tau' + 12\rho'\sigma']$$

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BH —> Type D —> $\Psi_3 = \Psi_4 = \kappa = \sigma = \kappa' = \sigma' = 0$

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Hence, linearising on type D leads to

$$\bar{\mathcal{O}}_4\delta\Psi_4 = 0$$

Penrose (1960)
Stewart & Walker (1974)
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Beyond linear order

Taking the second order variation of $\mathcal{O}_4\Psi_4 + \mathcal{O}_3\Psi_3 + \mathcal{O}_2\Psi_2 = 0$ leads to

$$\bar{\mathcal{O}}_4\delta^2\Psi_4 = -\,\delta\mathcal{O}_4\delta\Psi_4 - \delta\mathcal{O}_3\delta\Psi_3 \equiv \mathcal{S}[\delta g, \delta g]$$

Kegeles & Cohen (1979)
Chrzanowski (1976)
Wald (1978)
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It is not sufficient to know $\delta\Psi_4$, but we need to reconstruct δg_{ab}

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This is easy in vacuum (CCK, Wald…)

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Aksteiner, Andersson & Backdahl (2016)
Green, Hollands & Zimermann (2019)

Beyond linear order

Taking the second order variation of $\mathcal{O}_4\Psi_4 + \mathcal{O}_3\Psi_3 + \mathcal{O}_2\Psi_2 = 0$ leads to

$$\bar{\mathcal{O}}_4\delta^2\Psi_4 = -\delta\mathcal{O}_4\delta\Psi_4 - \delta\mathcal{O}_3\delta\Psi_3 \equiv \mathcal{S}[\delta g, \delta g]$$

It is not sufficient to know $\delta\Psi_4$, but we need to reconstruct δg_{ab}

This is easy in vacuum (CCK, Wald…)

This is not so easy in the presence of matter (but there's been recent progress!)

Kegeles & Cohen (1979)
Chrzanowski (1976)
Wald (1978)
Aksteiner, Andersson & Backdahl (2016)
Green, Hollands & Zimermann (2019)

IV. Quadratic QNMs

Oldboy (올드보이), 2003, Park Chan-Wook (박찬욱)

Teukolsky Master Equation

Solution to linear equation $\bar{\mathcal{O}}_4 \delta\Psi_4 = 0$

$$\delta\Psi_4 = \sum_{\ell,m,n,\sigma} \mathcal{A}_{\ell mn\sigma} e^{-i\omega_{\ell mn\sigma}(t-t_{\rm peak})} + \dots$$

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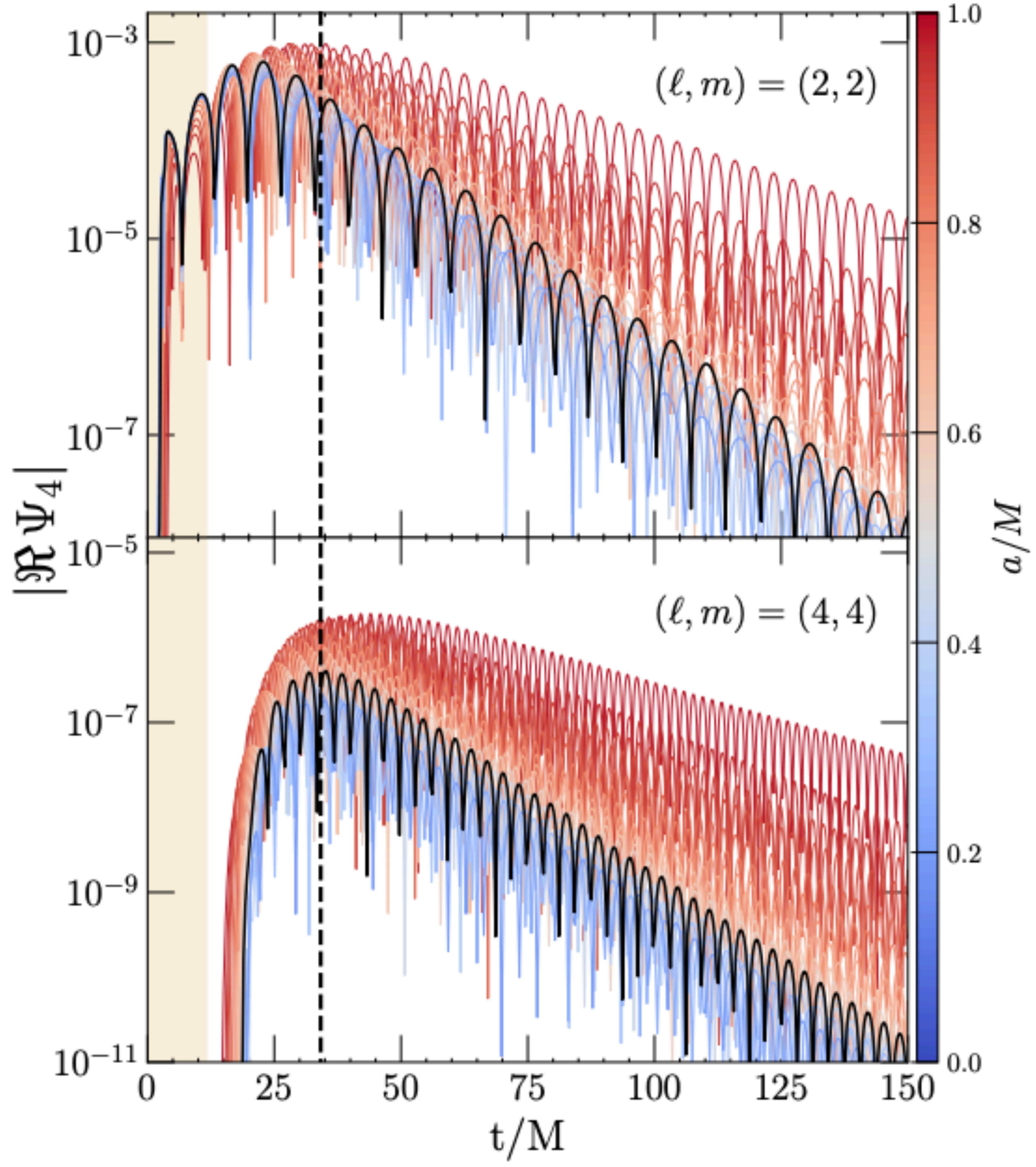
Inhomogeneous solution to second order has QQNMs!

$$\bar{\mathcal{O}}_4 \delta^2\Psi_4 = \mathcal{S}[\delta g, \delta g]$$

$$\delta^2\Psi_4 = \delta^2\Psi_4^{\rm hom} + \sum_{\lambda_1,\lambda_2} \mathcal{A}_{\lambda_1\times\lambda_2} e^{-i(\omega_{\lambda_1}+\omega_{\lambda_2})(t-t_{\rm peak})} + \dots$$



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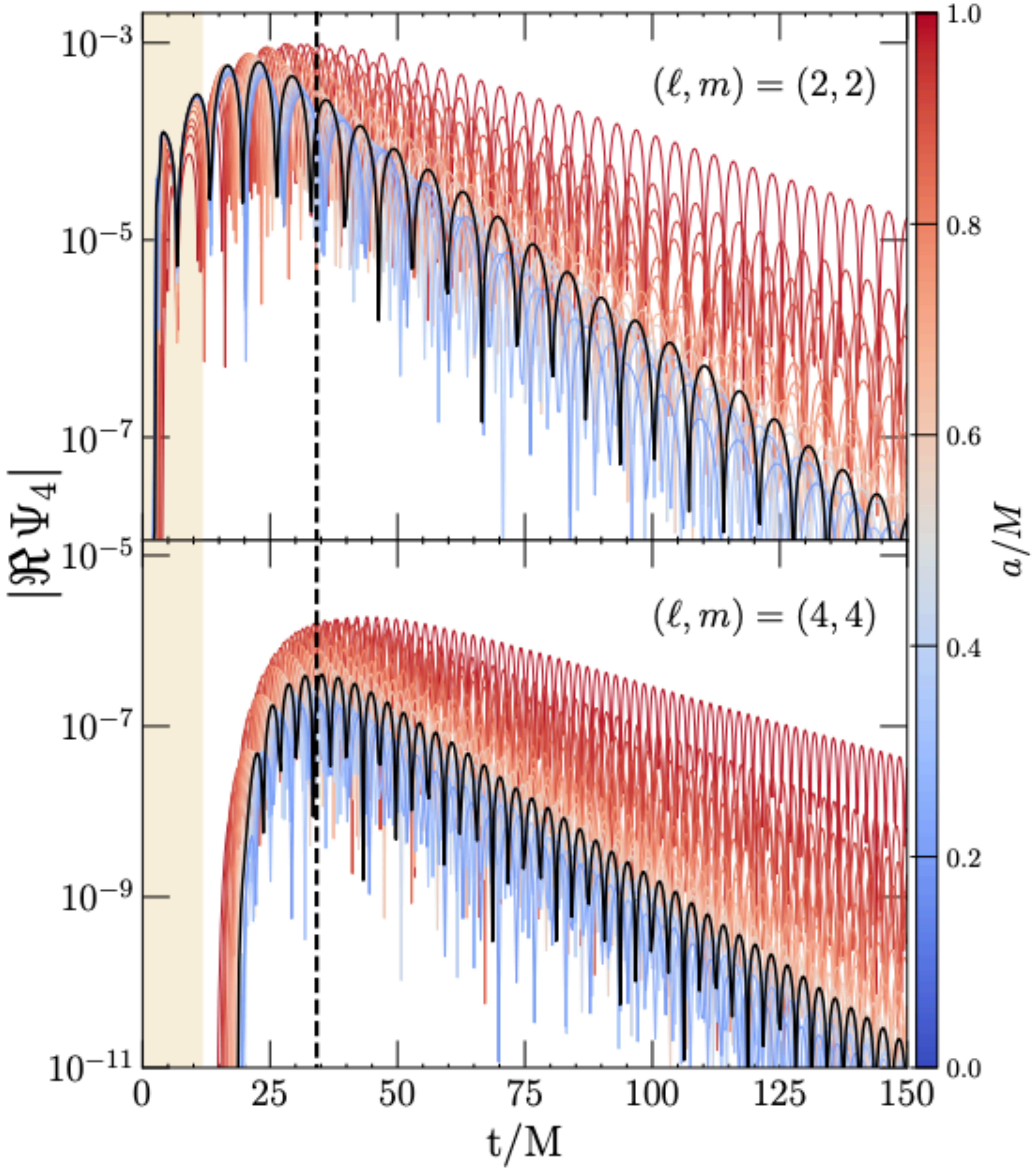
Coupling coefficient: $\mathcal{R} = \dfrac{\mathcal{A}_{\lambda_1\times\lambda_2}}{\mathcal{A}_{\lambda_1}\mathcal{A}_{\lambda_2}}$

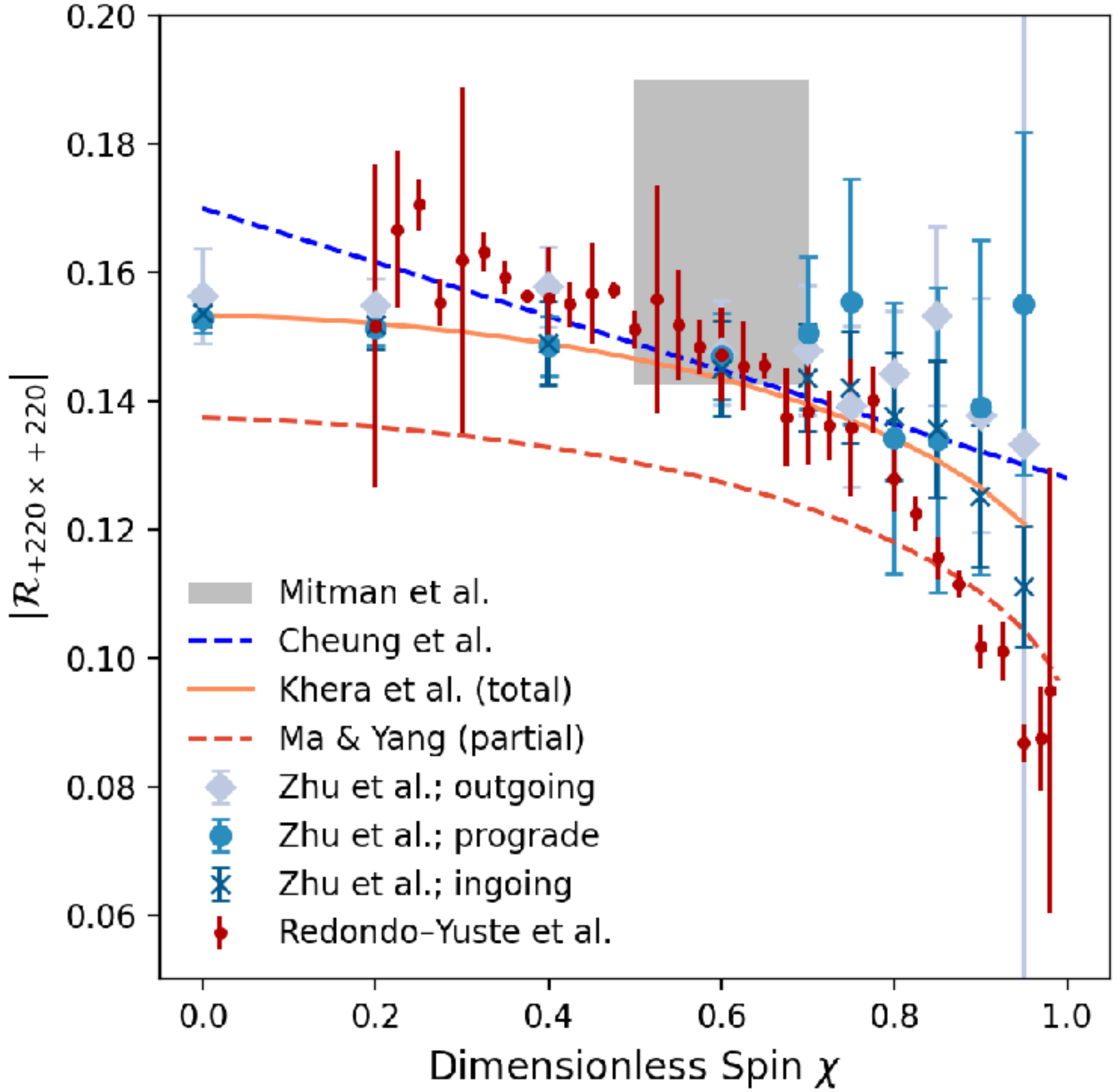


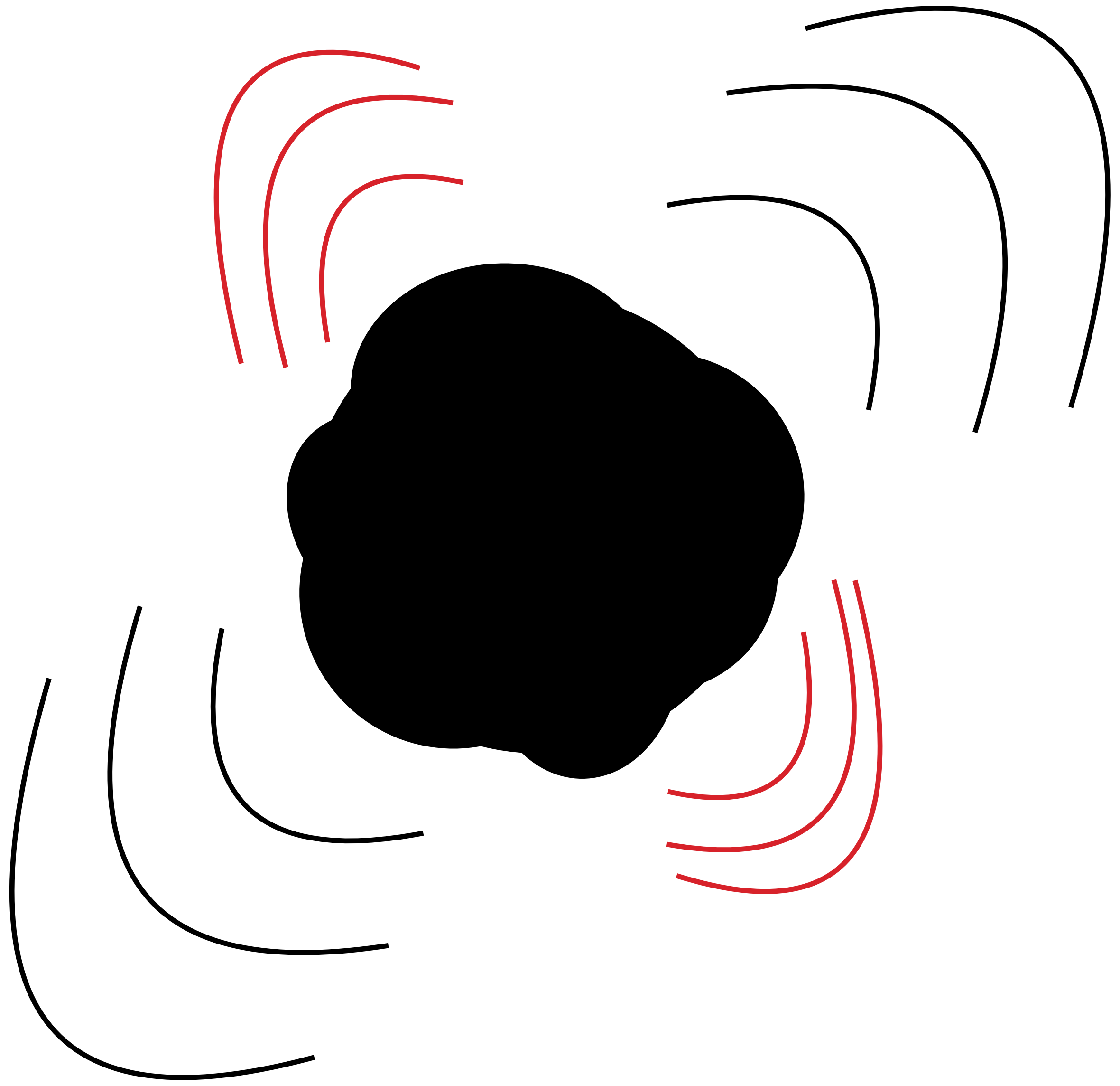
Campanelli & Lousto (1998)

JRY+ (2023)
Ripley+ (2021)

Black Hole Ringdown

Coupling coefficients depend only on (M, J)





Tiglio+ (1996)
Campanelli+ (1998)
Ioka+ (2007)
Brizuela+ (2008)
London+ (2014)
Cheung+ (2022)
Mitman+ (2022)
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Bourg+ (2024,25)
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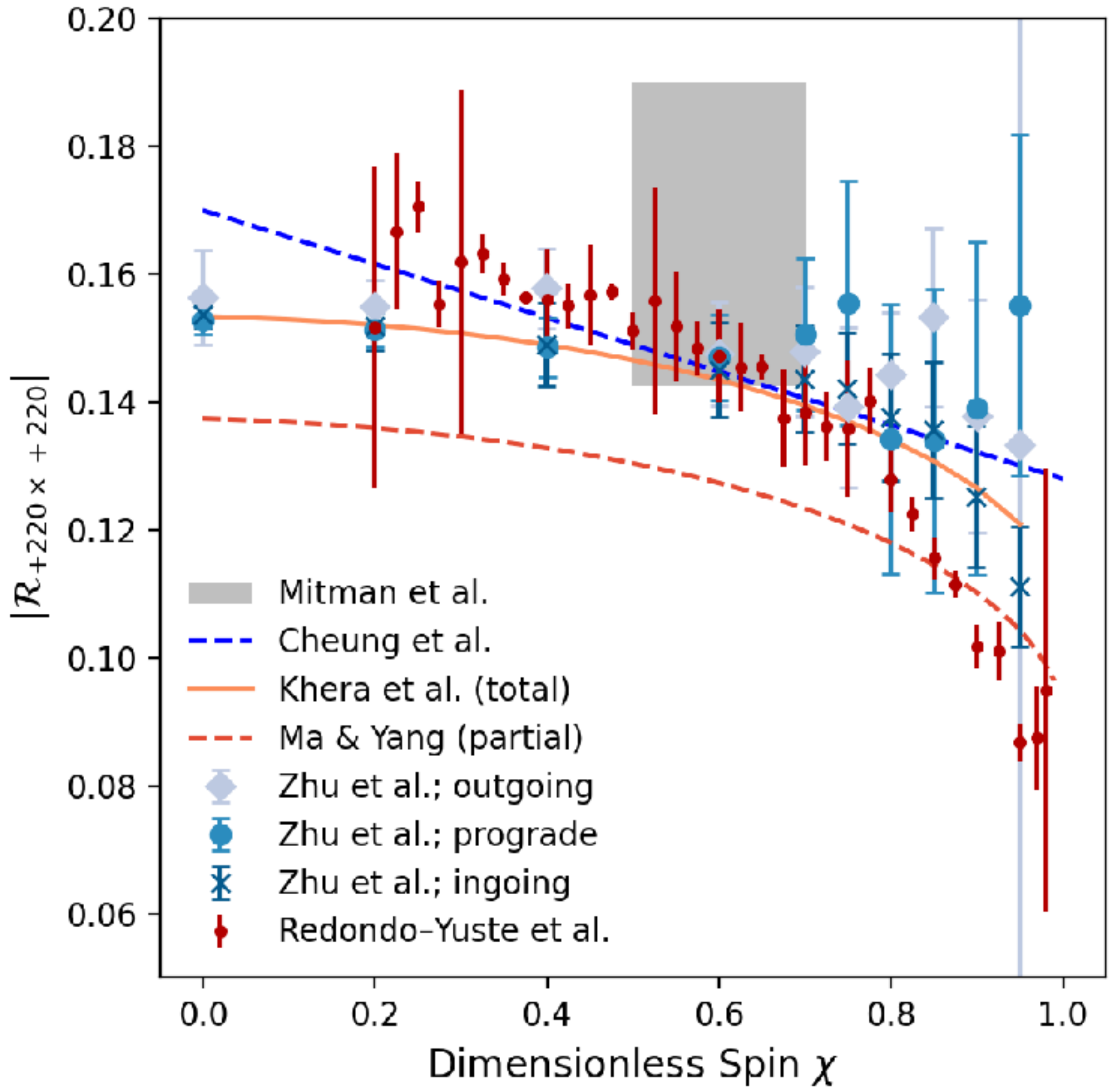

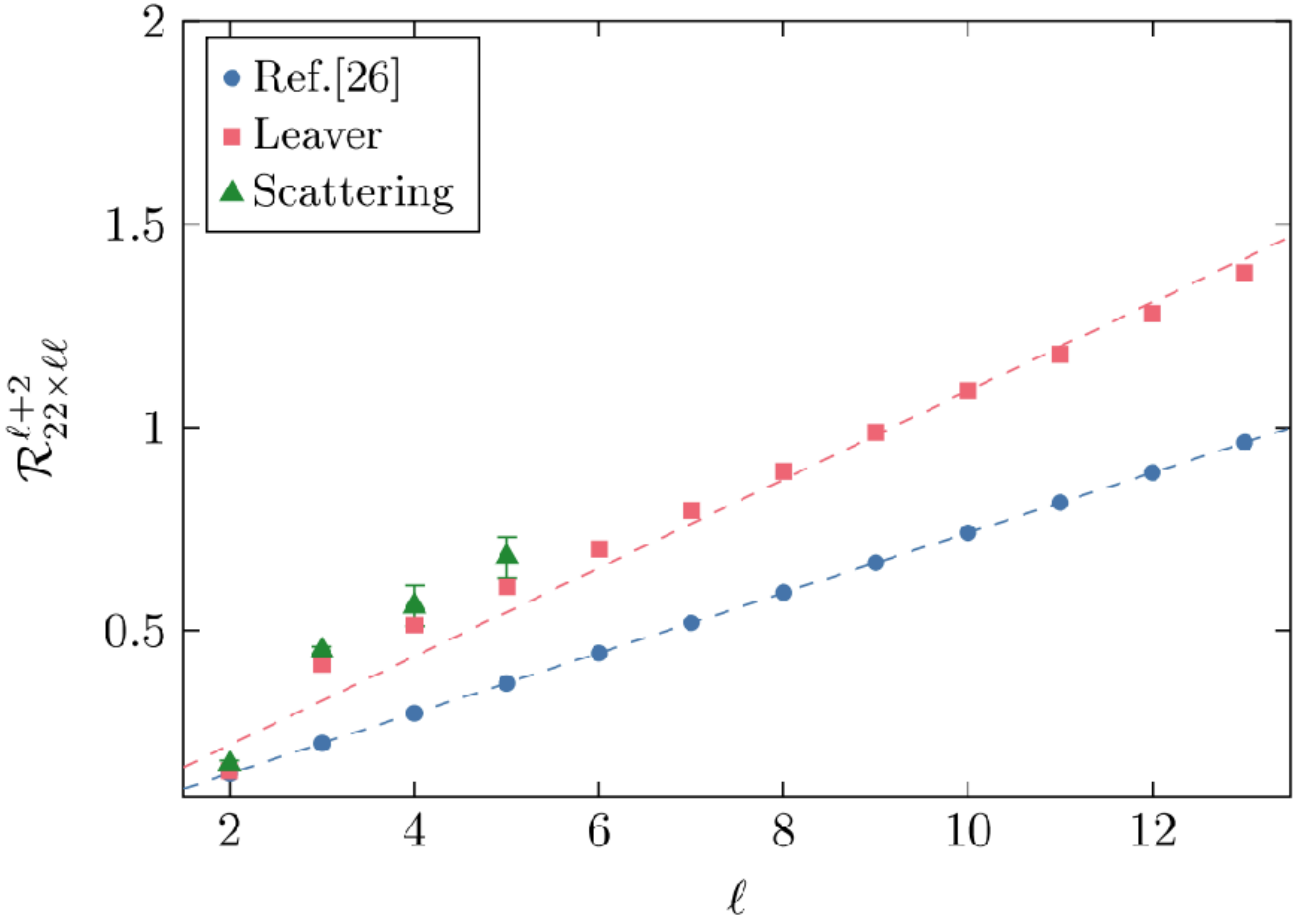

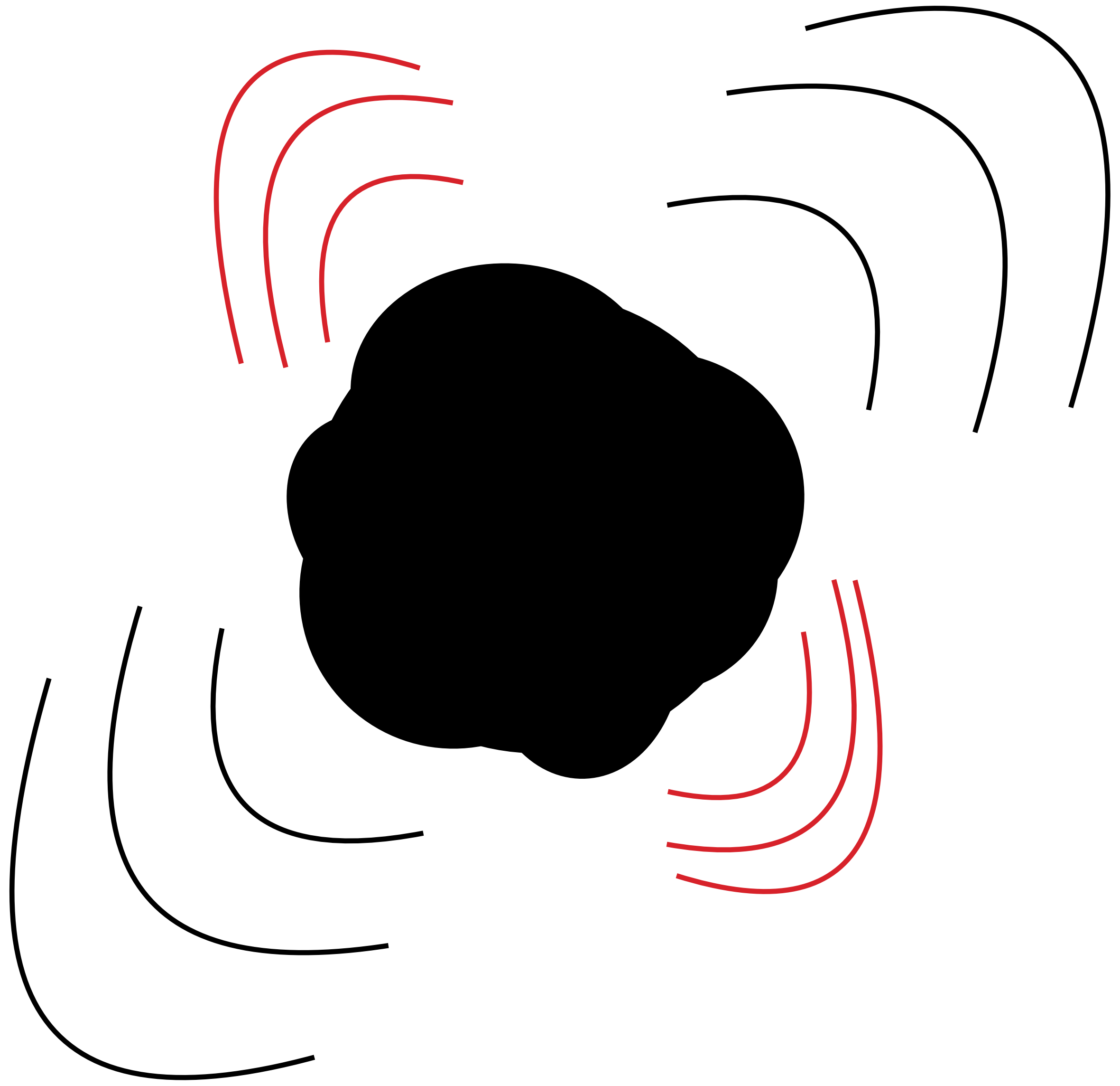

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+++

What happens at high frequencies?

V. Nonlinearities in plane waves

Decision to Leave (헤어질 결심),
2022, *Park Chan-Wook* (박찬욱)

Plane waves in gravity

Recall that the curvature satisfies a ***nonlinear wave equation***

$$\Box \Psi_{ABCD} = 6\Psi^{EF}_{(AB}\Psi_{CD)EF}$$

Penrose (1965)

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pp-waves are **exact solutions** where the RHS vanishes!

$$\Box \Psi_{ABCD} = 0$$

Penrose (1965)

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pp-waves are **exact solutions** where the RHS vanishes!

$$\Box \Psi_{ABCD} = 0$$

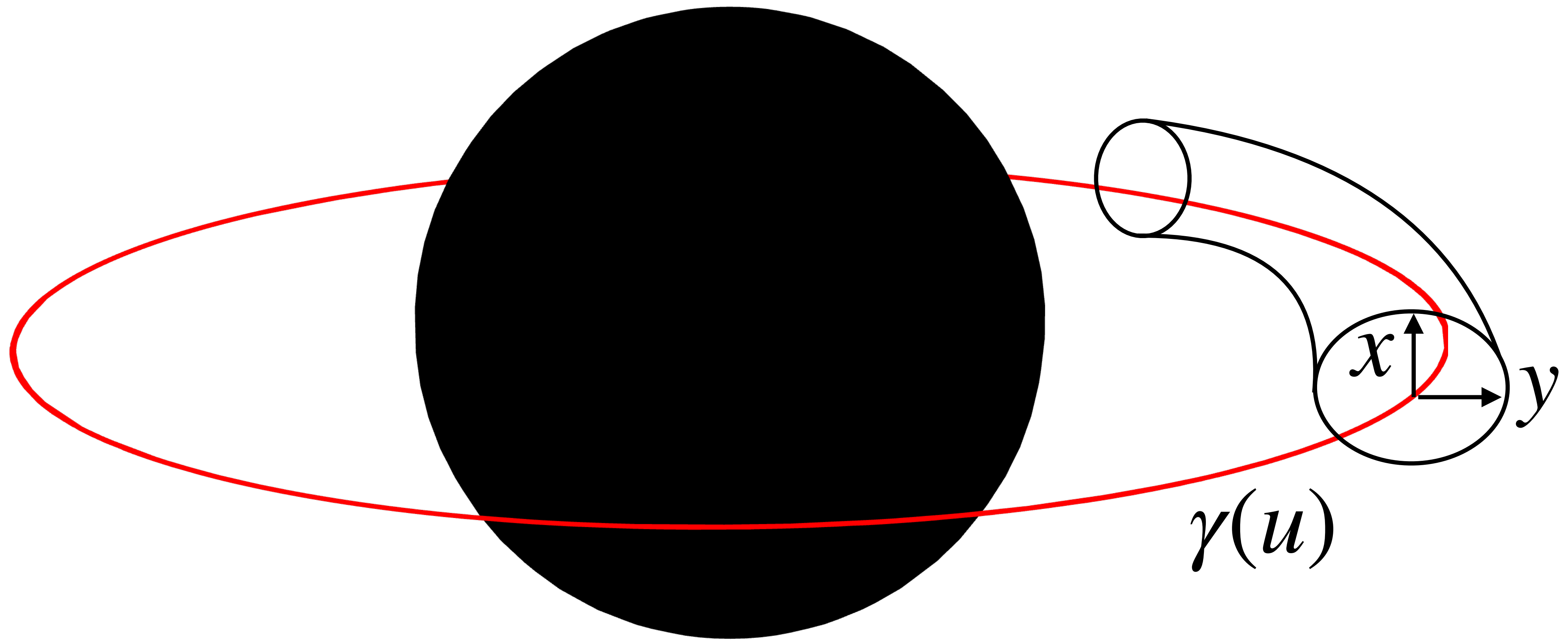
Plane waves are pp-waves which are symmetric

$$ds^2 = 2du dv - H(u)_{IJ} z^I z^J du^2 - dx^2 - dy^2 \,, \qquad z^I = (x, y)$$

Penrose (1965)

Zooming into a geodesic

Choosing coordinates adapted to a ***null geodesic***, and zooming in, spacetime becomes a plane wave



Penrose (1965)
Blau (2011)

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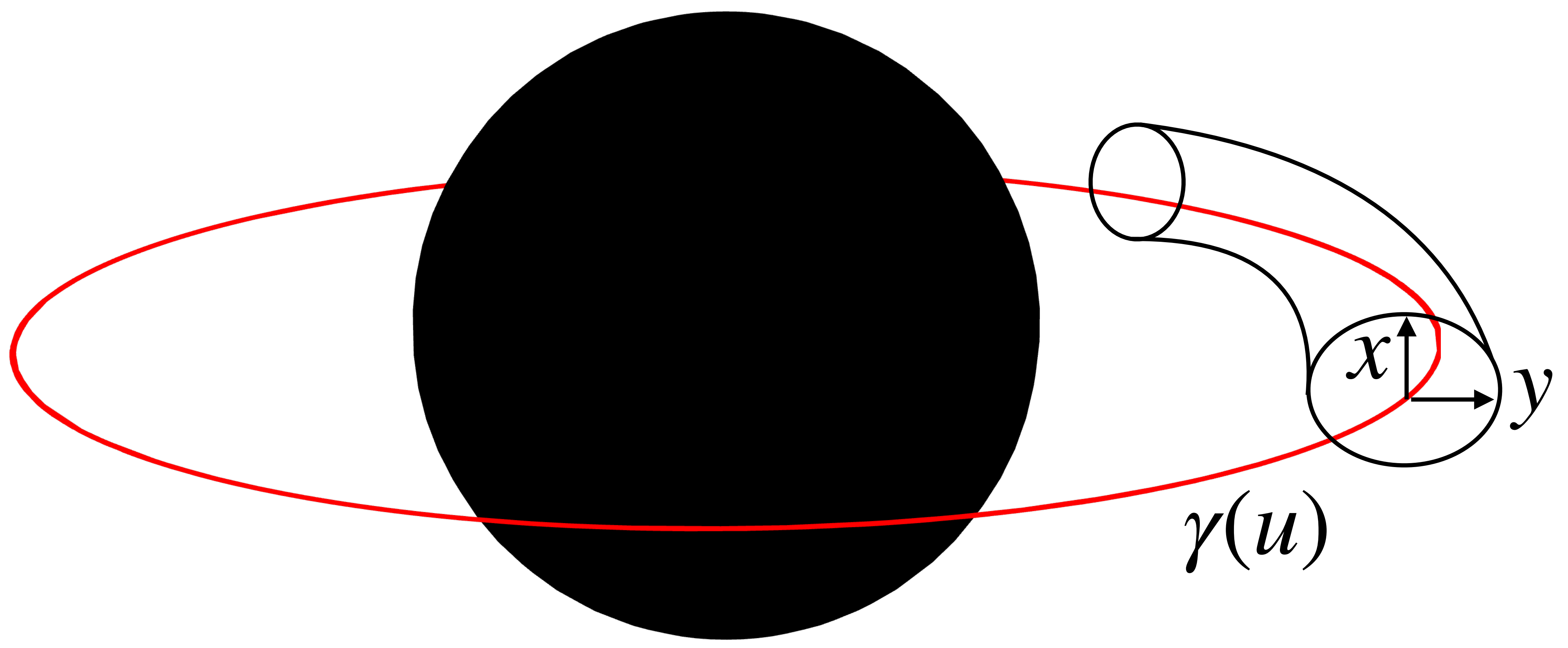

$$ds^2 = 2dudv - R_{IuJu}(u)z^I z^J du^2 - dx^2 - dy^2$$

Doing so at the (equatorial) ***light ring***, we find an homogeneous plane wave

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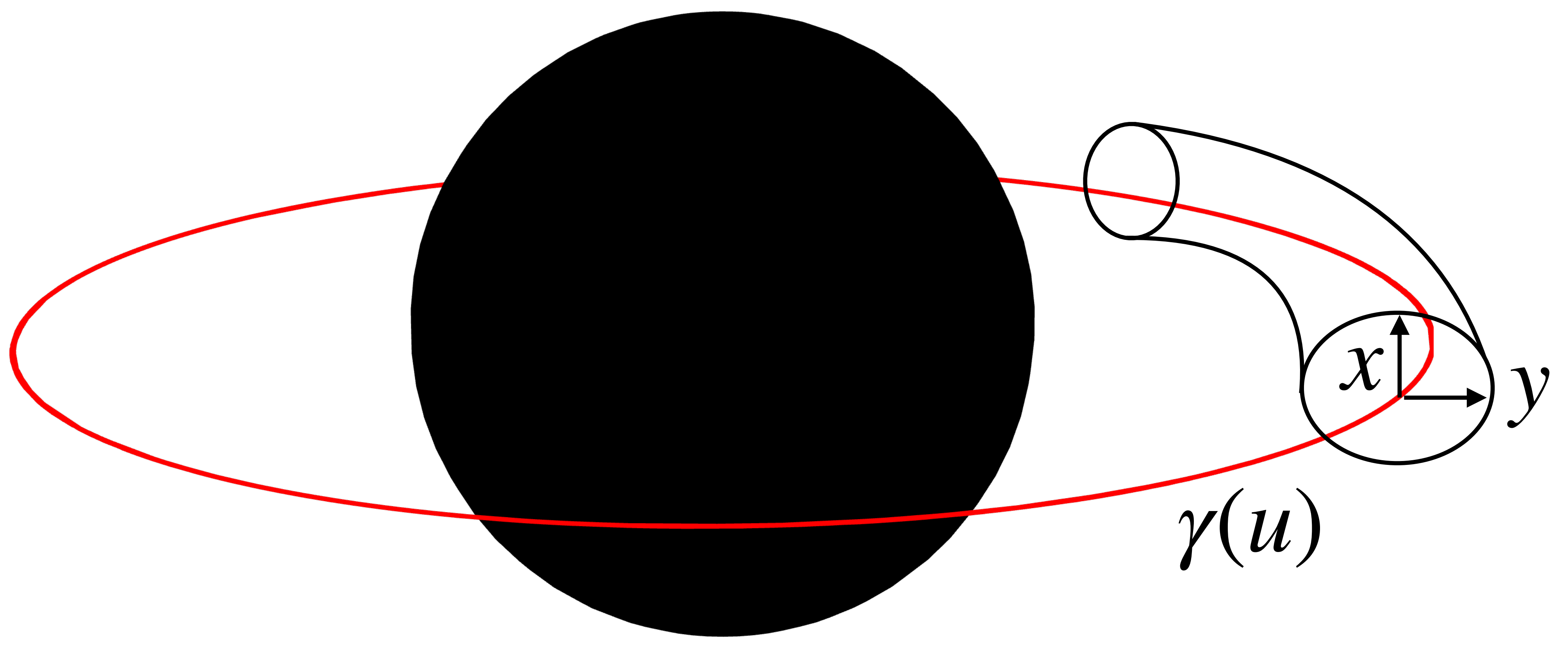

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Doing so at the (equatorial) ***light ring***, we find an homogeneous plane wave

$$ds^2 = 2dudv - \Omega_{\mathrm{LR}}^2(y^2 - x^2)du^2 - dx^2 - dy^2$$

Penrose (1965)
Blau (2011)

QNMs from plane waves

Linear perturbations around homogeneous plane wave

$$ds^2 = 2dudv - \Omega_{\rm LR}^2(y^2 - x^2)du^2 - dx^2 - dy^2 + \dot{g}_{ab}dx^a dx^b$$

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The previous derivation holds for Ψ_0 (aligned with the repeated PND)

$$\bar{\mathcal{O}}_0 \delta\Psi_0 = 0$$

Fransen (2023)

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The previous derivation holds for Ψ_0 (aligned with the repeated PND)

$$\bar{\mathcal{O}}_0 \delta\Psi_0 = 0$$

Solutions are analytical:

$$\delta\Psi_0 = e^{-ip_u u} e^{-ip_v v} e^{-|p_v|\Omega^2/2(x^2 - iy^2)} H_{n_x}\left(\sqrt{\Omega\,|\,p_v\,|}\,x\right) H_{n_y}\left(\sqrt{-i\Omega p_v}\,y\right)$$

Fransen (2023)

QNMs from plane waves

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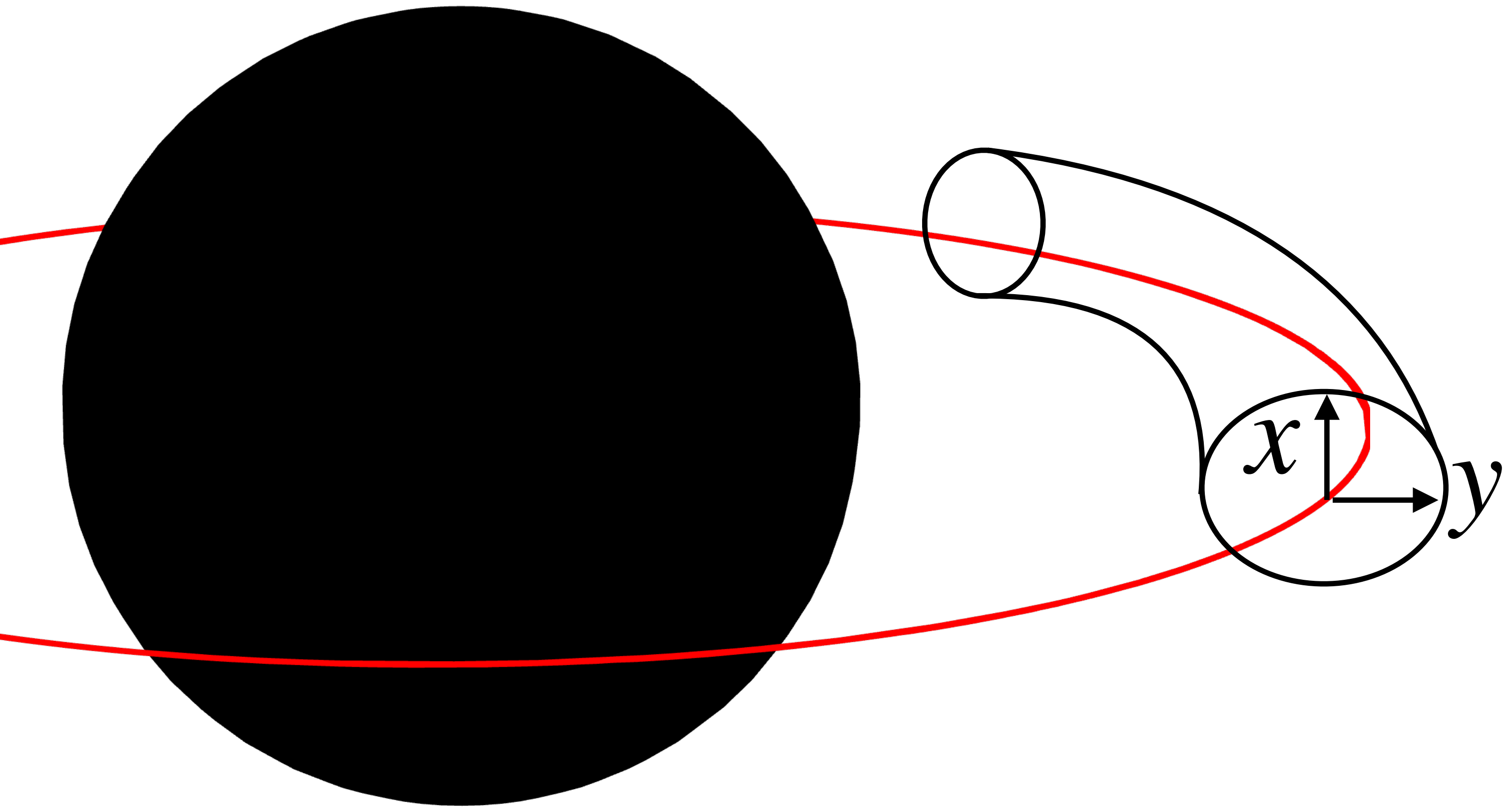

Fransen (2023)

QNMs from plane waves

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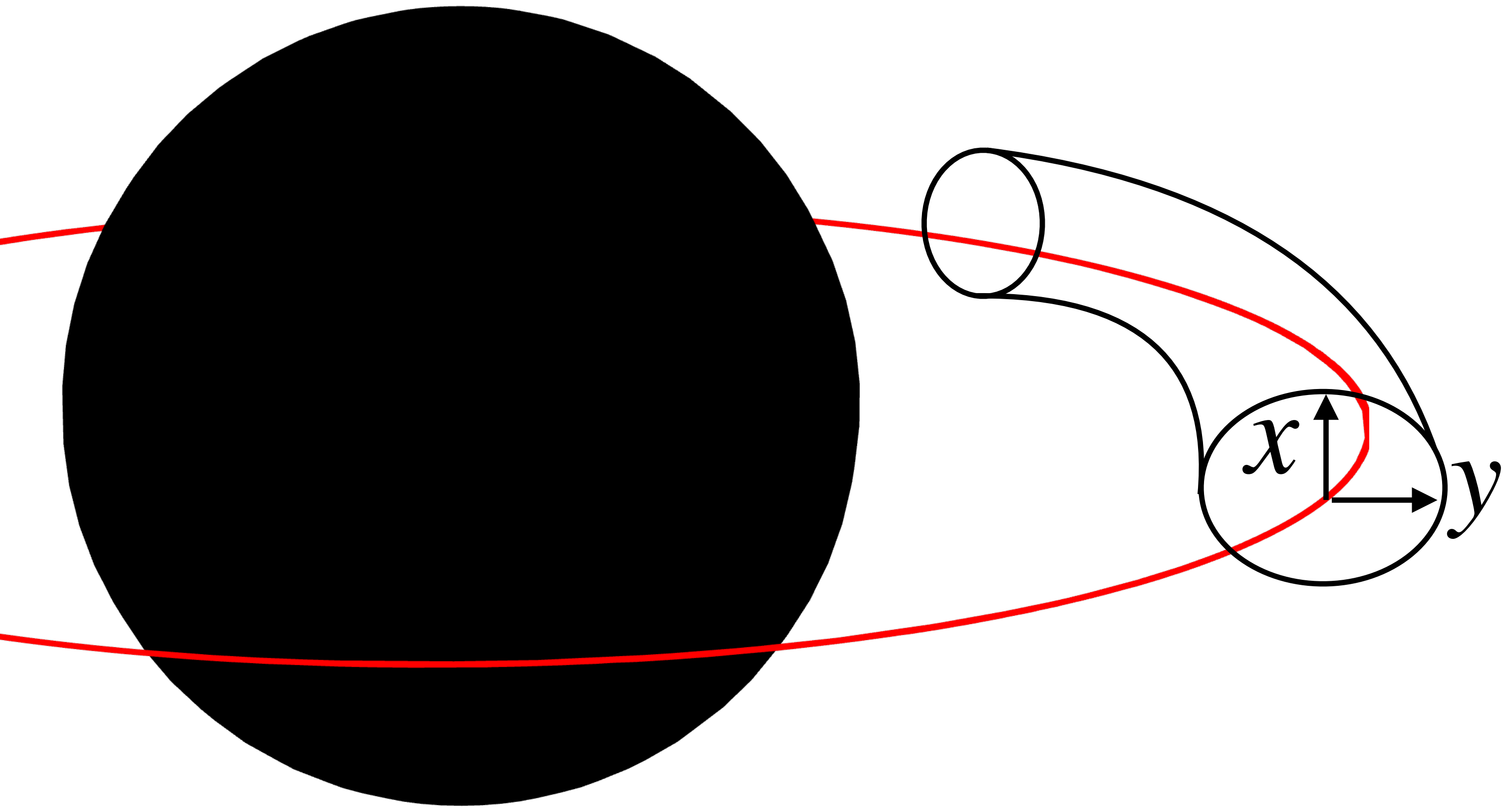

Impose boundary conditions:

- Periodic in the x direction
- Unstable in the y direction

QNMs from plane waves

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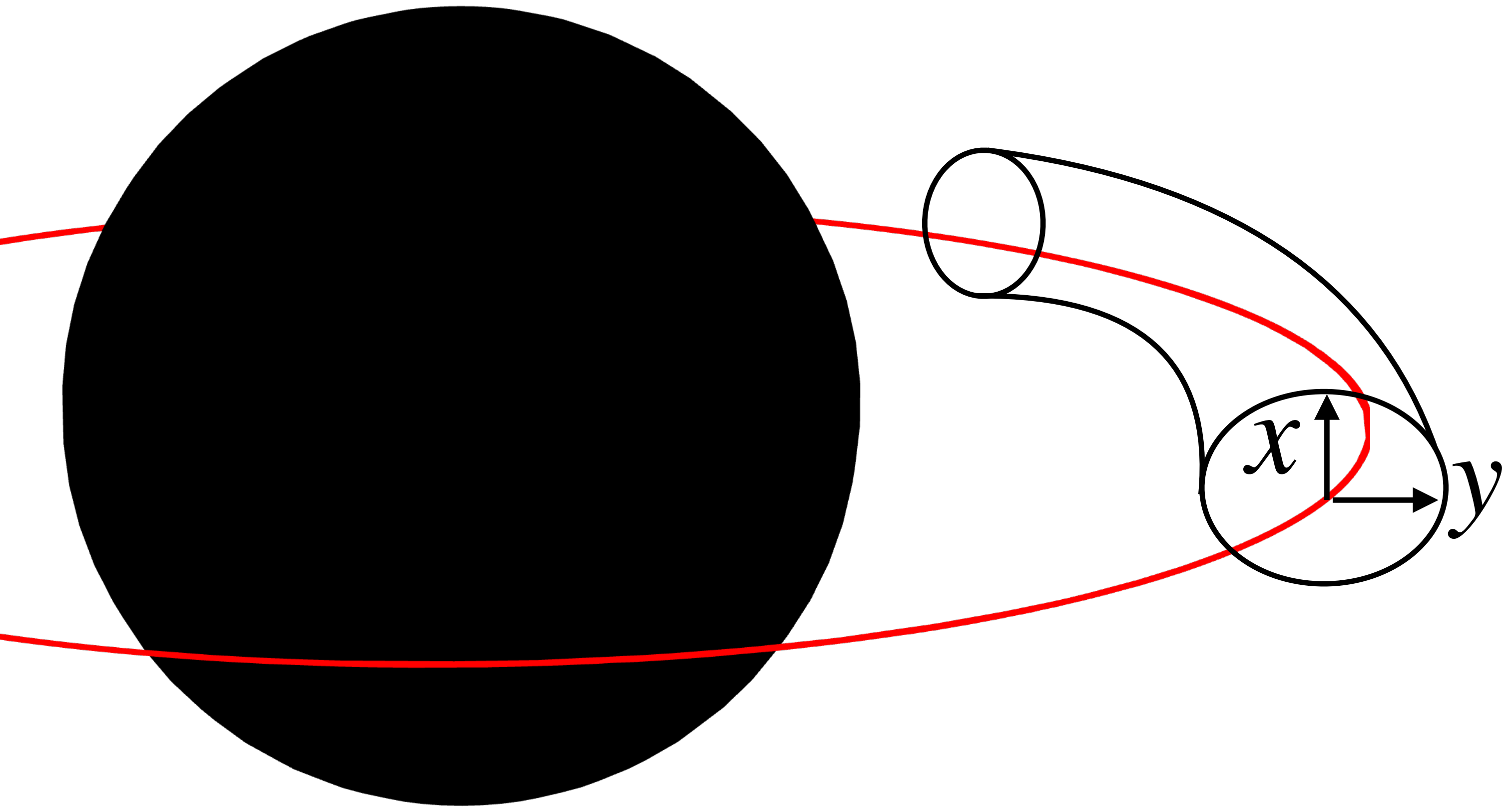

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$$\Omega^{-1} p_u = n_x + \frac{1}{2} + i\left(n_y + \frac{1}{2}\right)$$

Fransen (2023)

QNMs from plane waves

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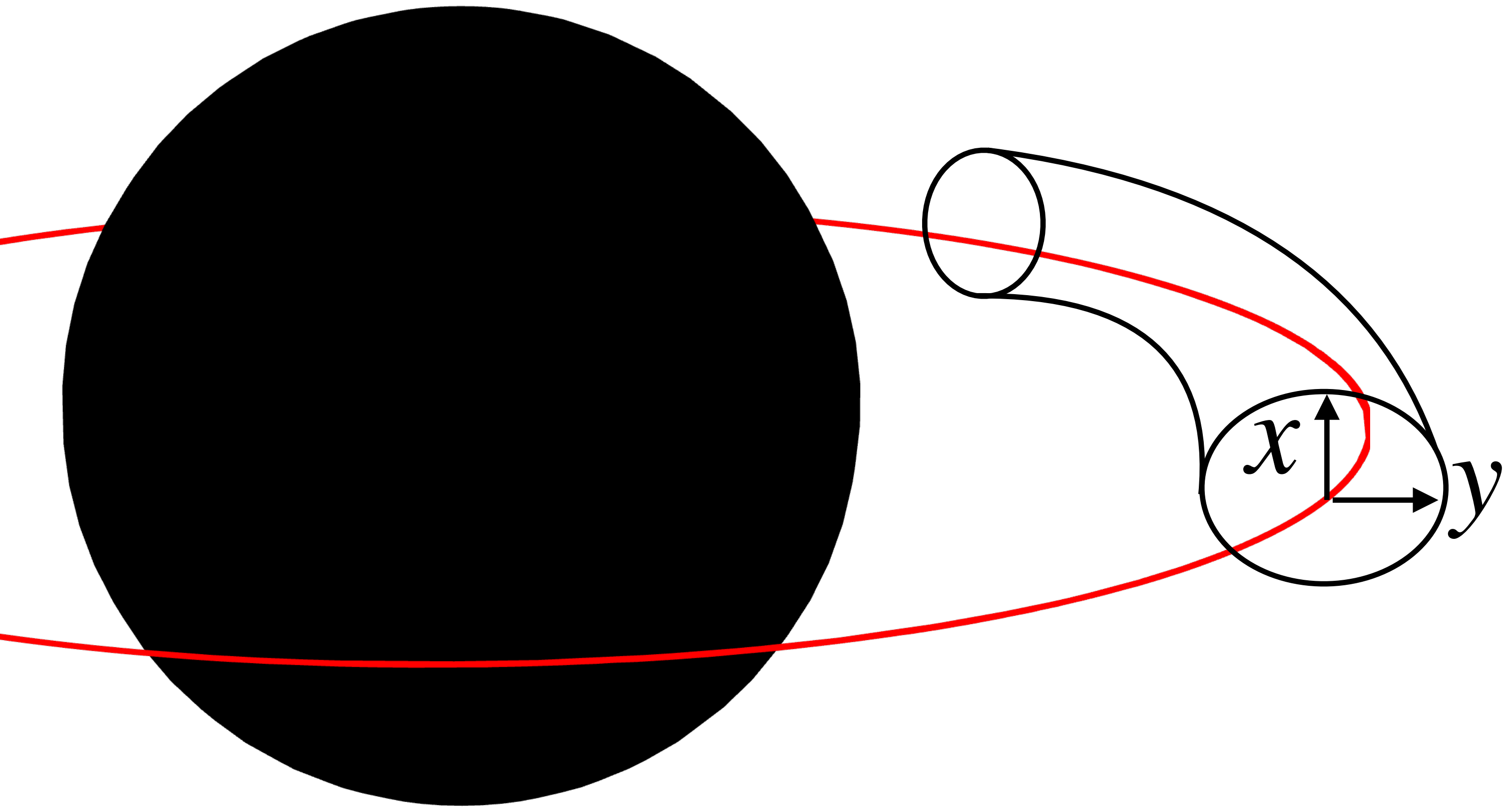

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Identify the frequencies of a Kerr BH in the ***high frequency limit***

$$p_u = \omega_{\ell mn}\,,p_v = m\Omega\,,n_x = \ell - |\,m\,|\,,n_y = n$$

Fransen (2023)

QQNMs from plane waves

Carry the same calculation up to second order, $\bar{\mathcal{O}}_0 \delta^2 \Psi_0 = \mathcal{S}[\delta g, \delta g]$

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Teukolsky—Starobinsky identities in (homogeneous) plane waves

$$\dot{\Psi}_i = -\frac{1}{2}\eth^i Þ^{4-i}\bar{\Psi}_H \qquad (i = 0,\ldots,4)$$

$$Þ^m \eth'^{4-m}\dot{\Psi}_n = Þ^{4-n}\eth'^n\dot{\Psi}_{4-m}\,, \qquad (0 \leq m, n \leq 4)$$

Fransen, Pereñiguez & JRY **(upcoming!)**

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Source term given uniquely in terms of Hertz potential

*Fransen, Pereñiguez & JRY **(upcoming!)***

QQNMs from plane waves

Particular solution to the 2nd order Teukolsky equation can be found analytically!

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$\delta^2\Psi_0$ is well-defined, and measures curvature fluctuations at the lightring!

$$\delta^2\Psi_0 = (\delta^2\Psi_0)_{\text{hom.}} + \sum_{\lambda_1,\lambda_2,\lambda} \mathcal{A}^{\lambda}_{\lambda_1\times\lambda_2}\psi_\lambda\,, \qquad \lambda = (p_u, p_v, n_x, n_y)$$

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Define coupling coefficients / nonlinear ratios (at the lightring)

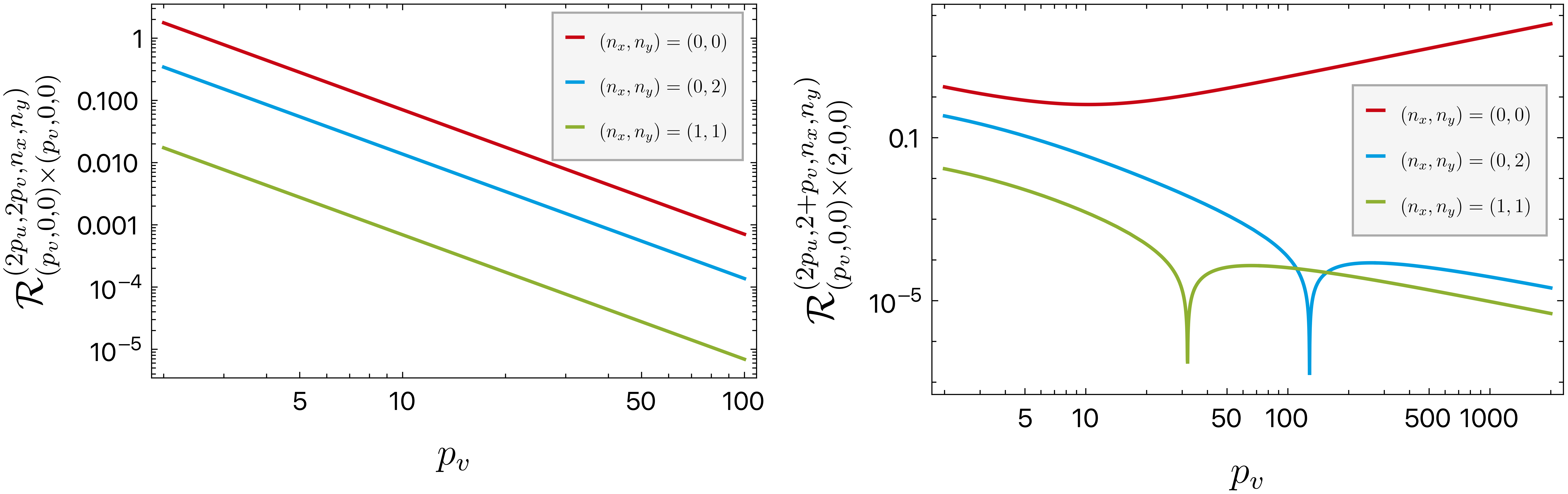
$$\mathcal{R}^{\lambda}_{\lambda_1\times\lambda_2} = \frac{\mathcal{A}^{\lambda}_{\lambda_1\times\lambda_2}}{\mathcal{A}_{\lambda_1}\mathcal{A}_{\lambda_2}}$$

Fransen, Pereñiguez & JRY (upcoming!)

QQNMs from plane waves

Ratios can be computed analytically! Stay tuned for results

$$\mathscr{R}^{(2p_u, p_v+\tilde{p}_v, 0,0)}_{(p_v,0,0)\times(\tilde{p}_v,0,0)} = \frac{ip_v}{8\tilde{p}_v^3} + \frac{57i}{16 p_v \tilde{p}_v} - \frac{9i}{8p_v^2} + \mathscr{O}(p_v^{-2})$$



Fransen, Pereñiguez & JRY **(upcoming!)**

VI. Conclusions

Conclusions

- Studying the back-reaction of perturbations is important in a number of contexts
- Important theoretical & mathematical consequences (stability of AdS, Kerr…)
- Non-linear ***dynamical*** effects will be observed (SIGW, QQNMS, memory…)
- New methods are needed to push ***beyond vacuum second order***
- Type N (Homogeneous ***plane waves***) might shed light on high frequency GWs
- Ultimate goal: ***Black Hole*** spacetimes + realistic ***matter*** fields