perturbation theory is unreasonably effective

until it isn't

Jaime Redondo-Yuste

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based on 2603.04501 & 2607.XXXXX

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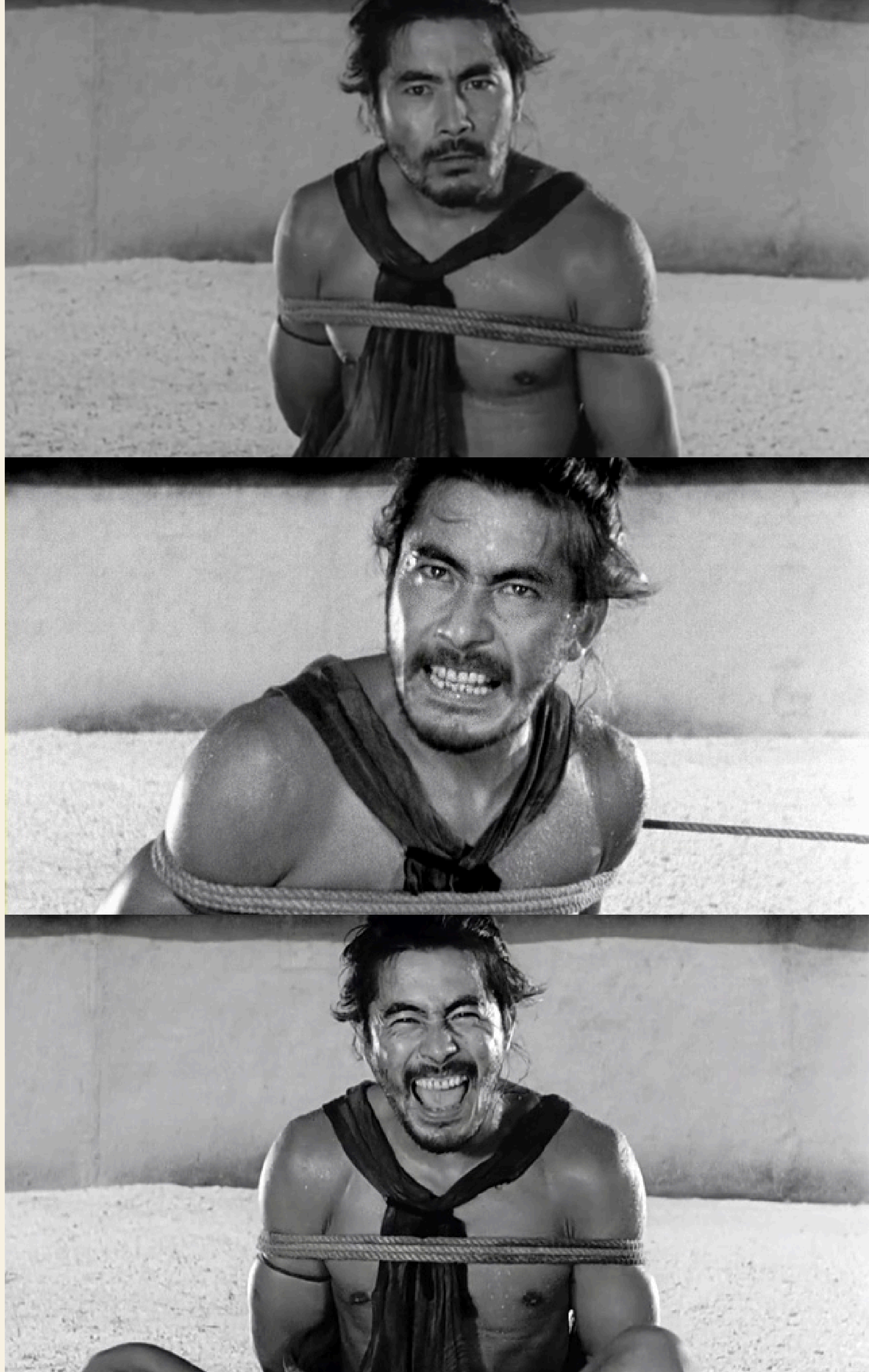


Outline

I Where perturbation theory works, and where it (probably) doesn't

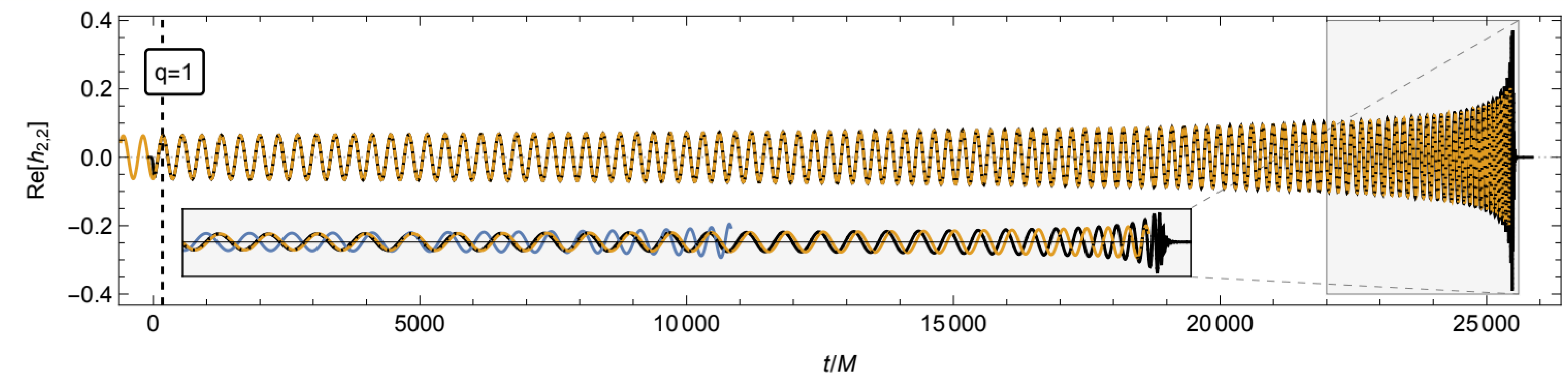
II Breakdown of perturbation theory at the edge of black hole formation

III Scattering waves off black holes

I The “unreasonable” success

The 2-body problem

Expanding the two-body problem in $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ is very accurate, even as $\nu \to \frac{1}{4}$.

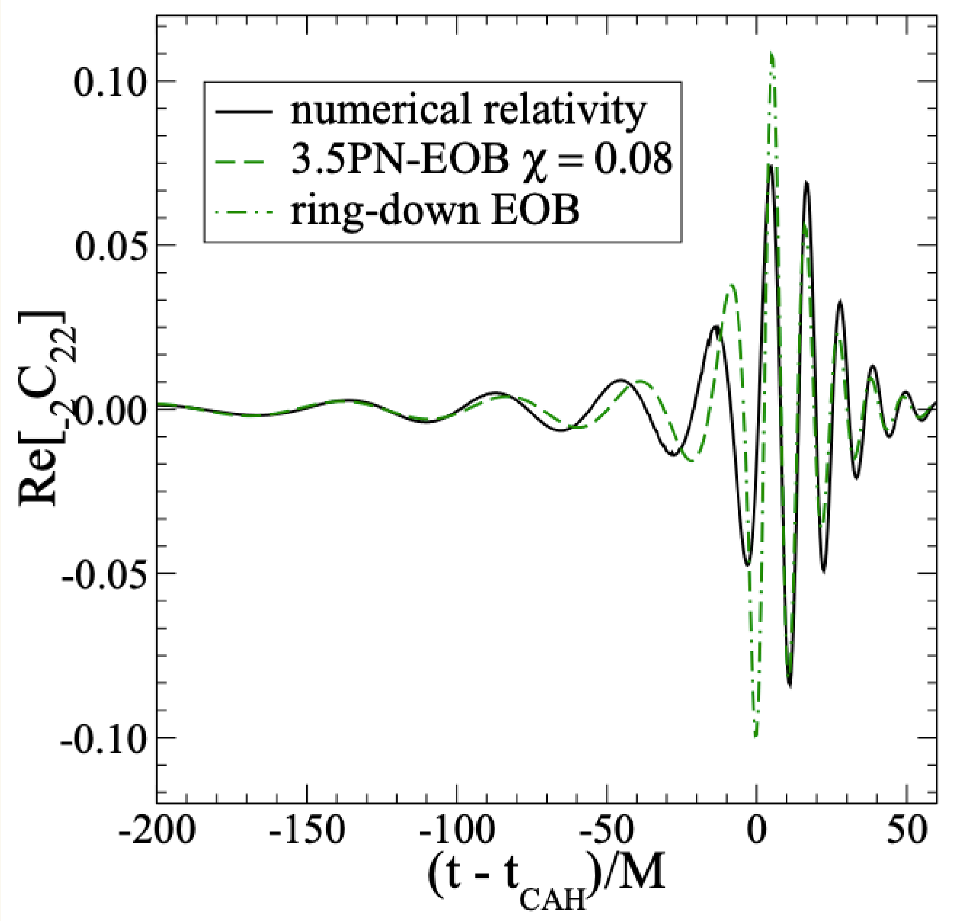


Dephasing is $\Delta\varphi \approx 1$ rad until the last 10 orbits.

[Wardell+ (2023)]

The 2-body problem

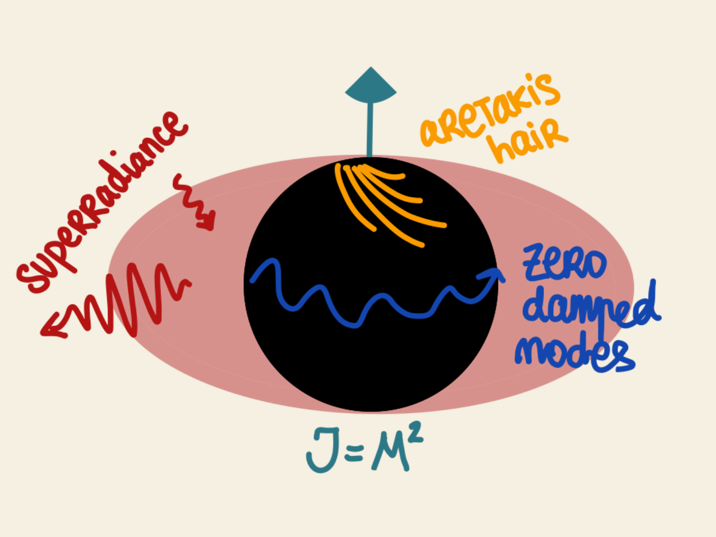
- Inspiral: (resummations of the) post-Newtonian expansion in $\frac{v}{c}$.
- Ringdown: quasinormal modes – poles of $(\delta G_{ab})^{-1}$ on a BH background.
- IMR: Smoothly join the two regimes.



[Buonanno-Cook-Pretorius (2007)]

Known limits of perturbation theory

- **Extreme spins**



[East & Pretorius (2013)]

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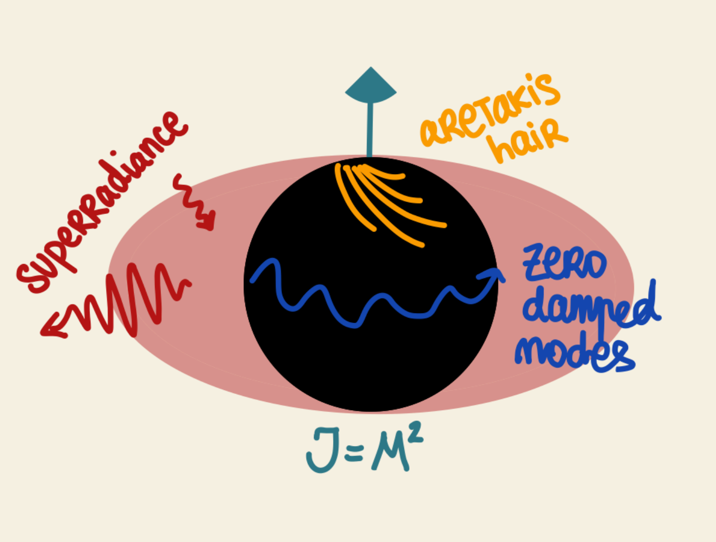
- High velocities



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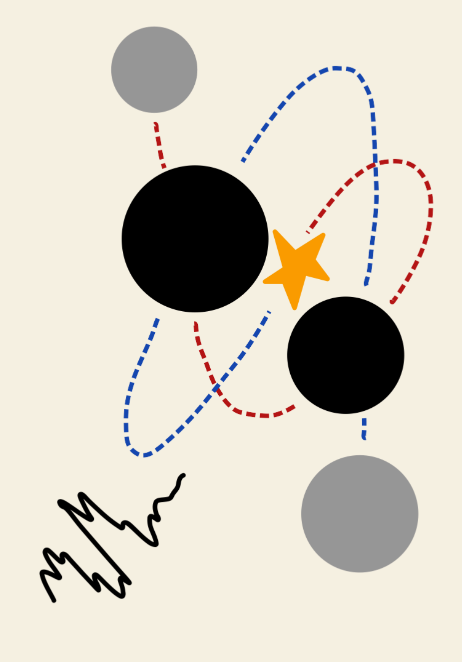

[Zhu+ (2026)]

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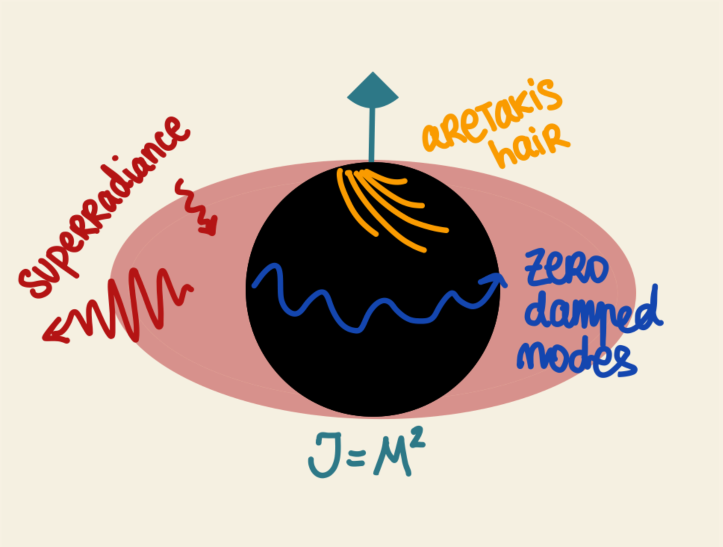

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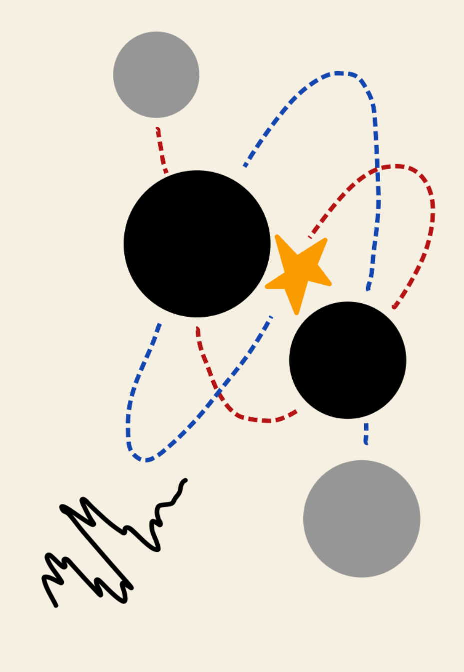
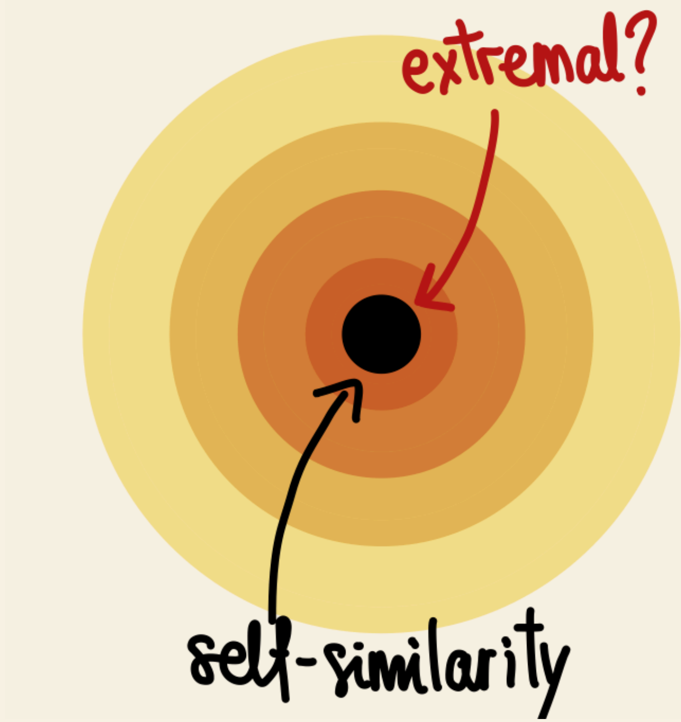


[Zhu+ (2026)]

- **Black Hole Formation**



[Choptuik (1993)]

II Black Hole Formation

in collaboration with

Josu Aurrekoetxea

Einstein–Klein–Gordon in spherical symmetry

Collapse of a massless scalar field Φ in spherical symmetry

$$G_{ab} = 8\pi\left(\nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}\nabla_c\Phi\nabla^c\Phi\right),$$

$$\Box_g\Phi = 0$$

[Christodoulou (1986), Choptuik (1993)]

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$\varepsilon = 1$ is the threshold between dispersion and BH formation.

[Christodoulou (1986), Choptuik (1993)]

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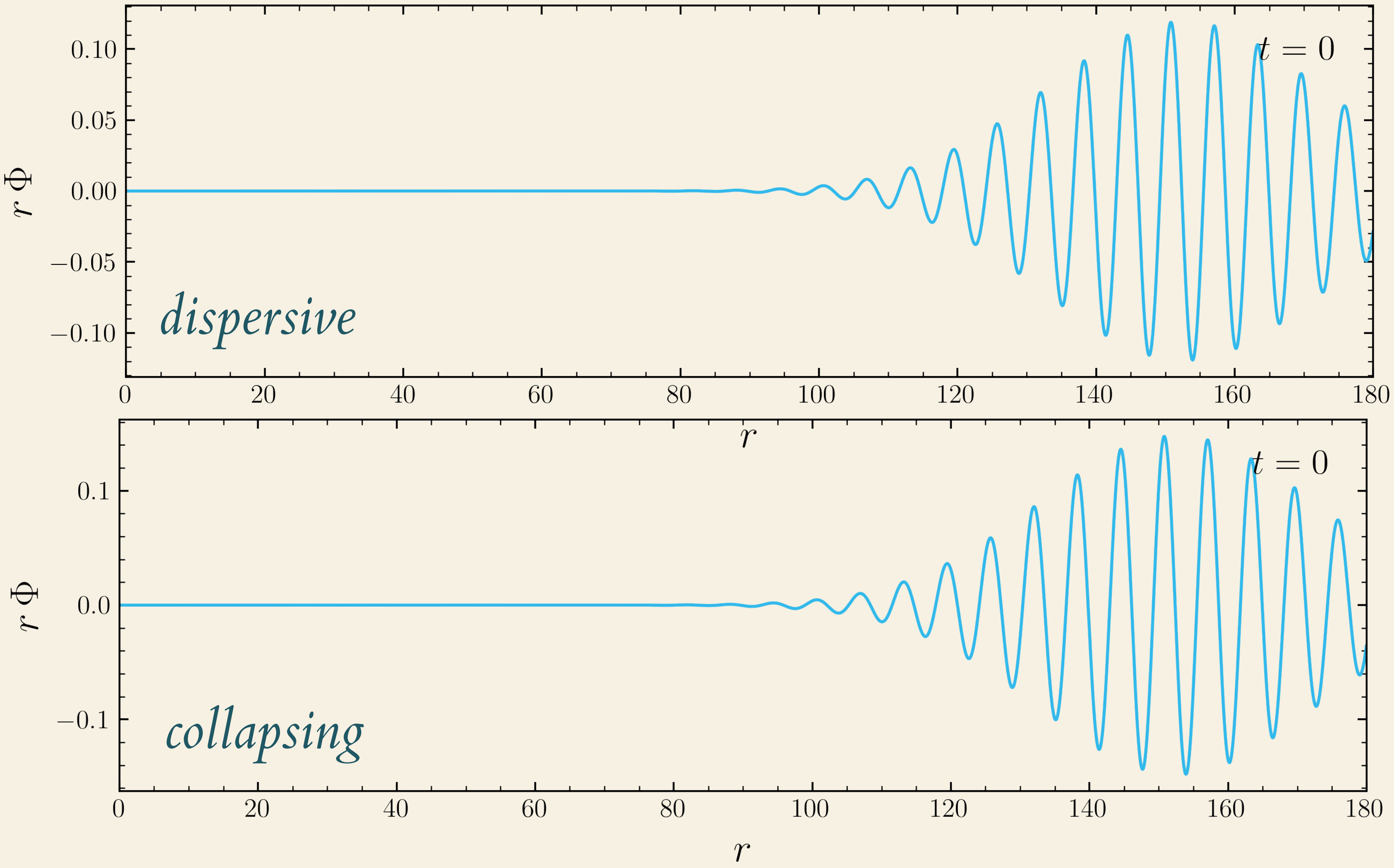

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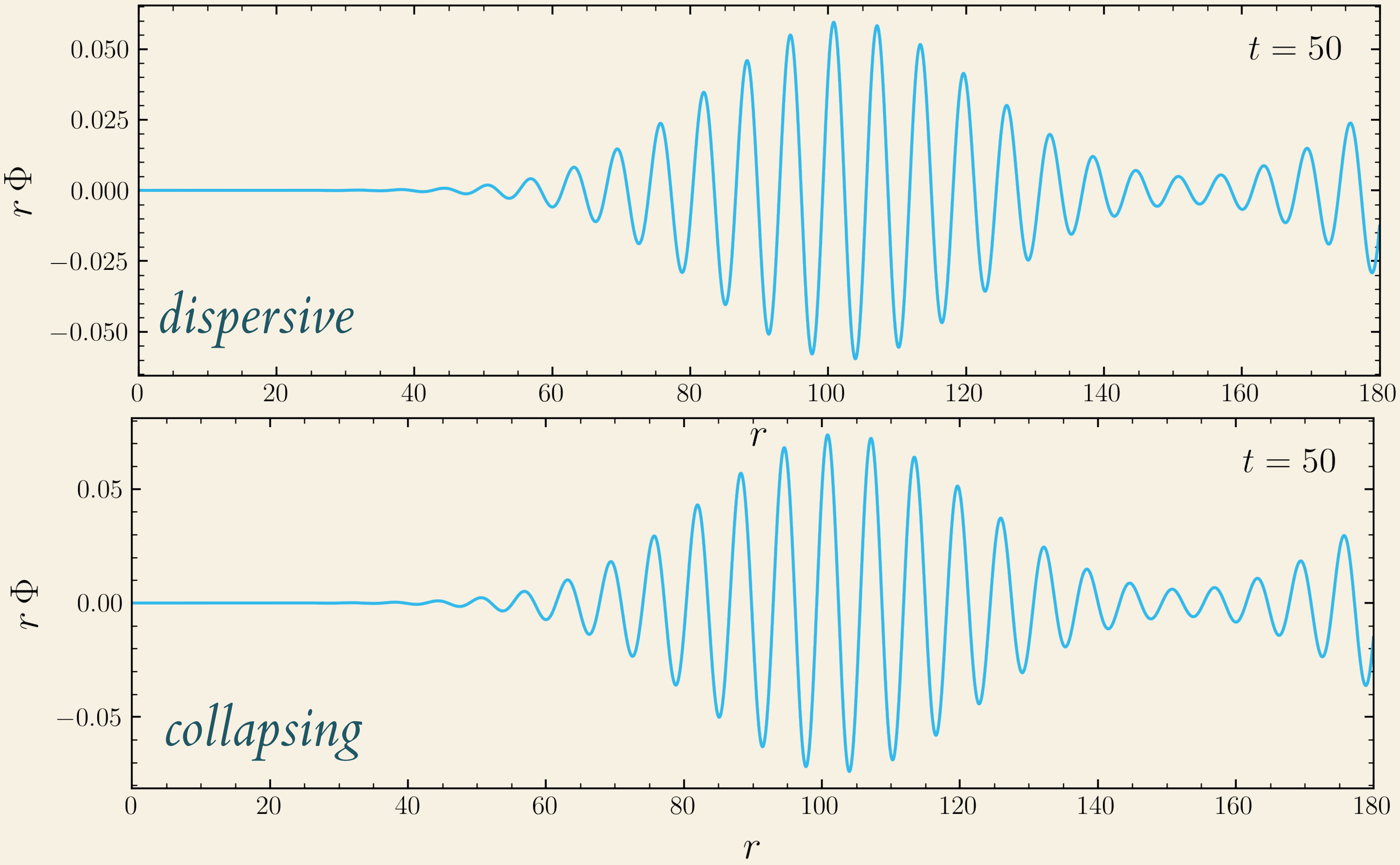

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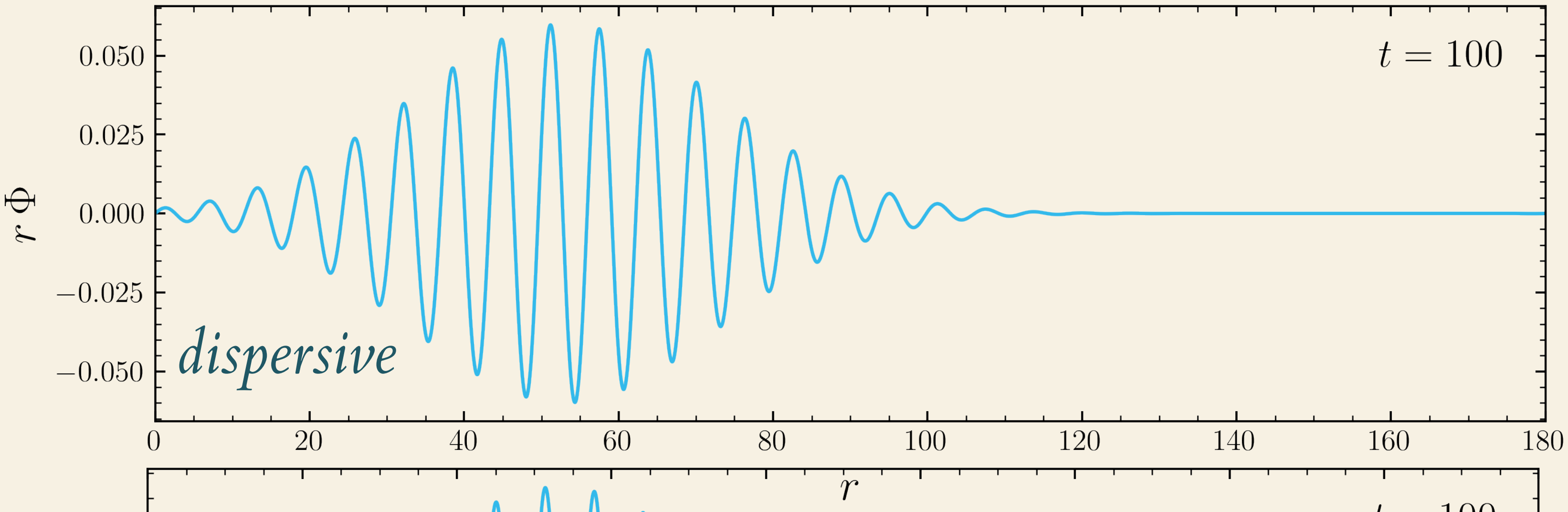

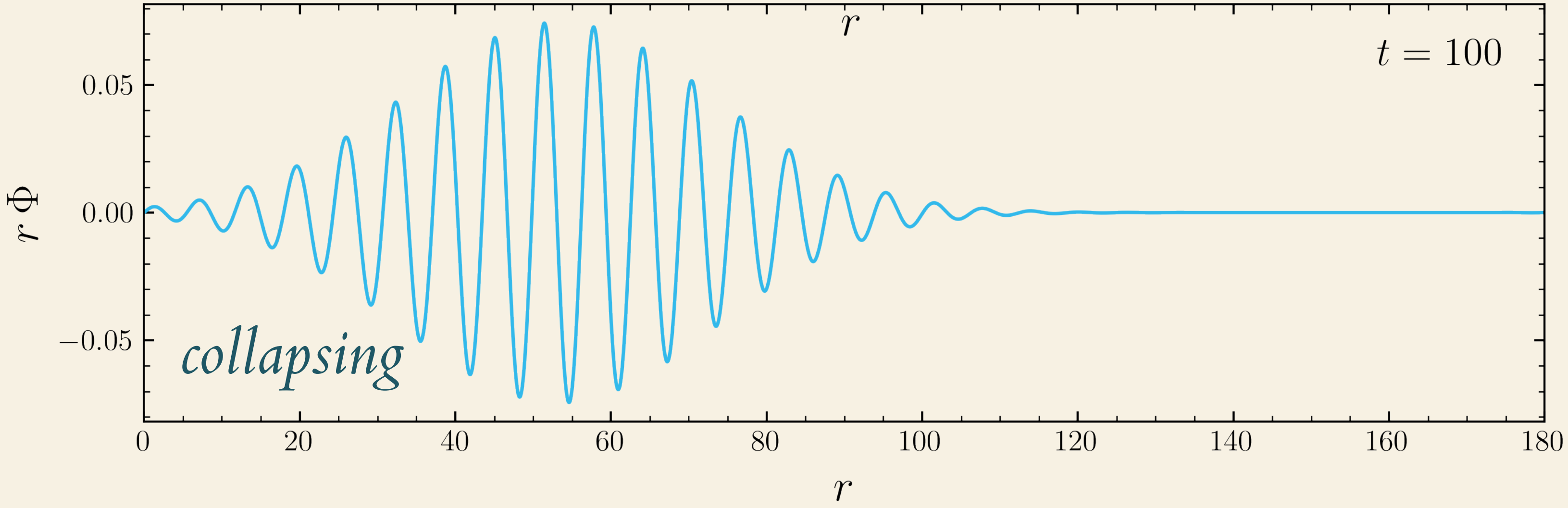

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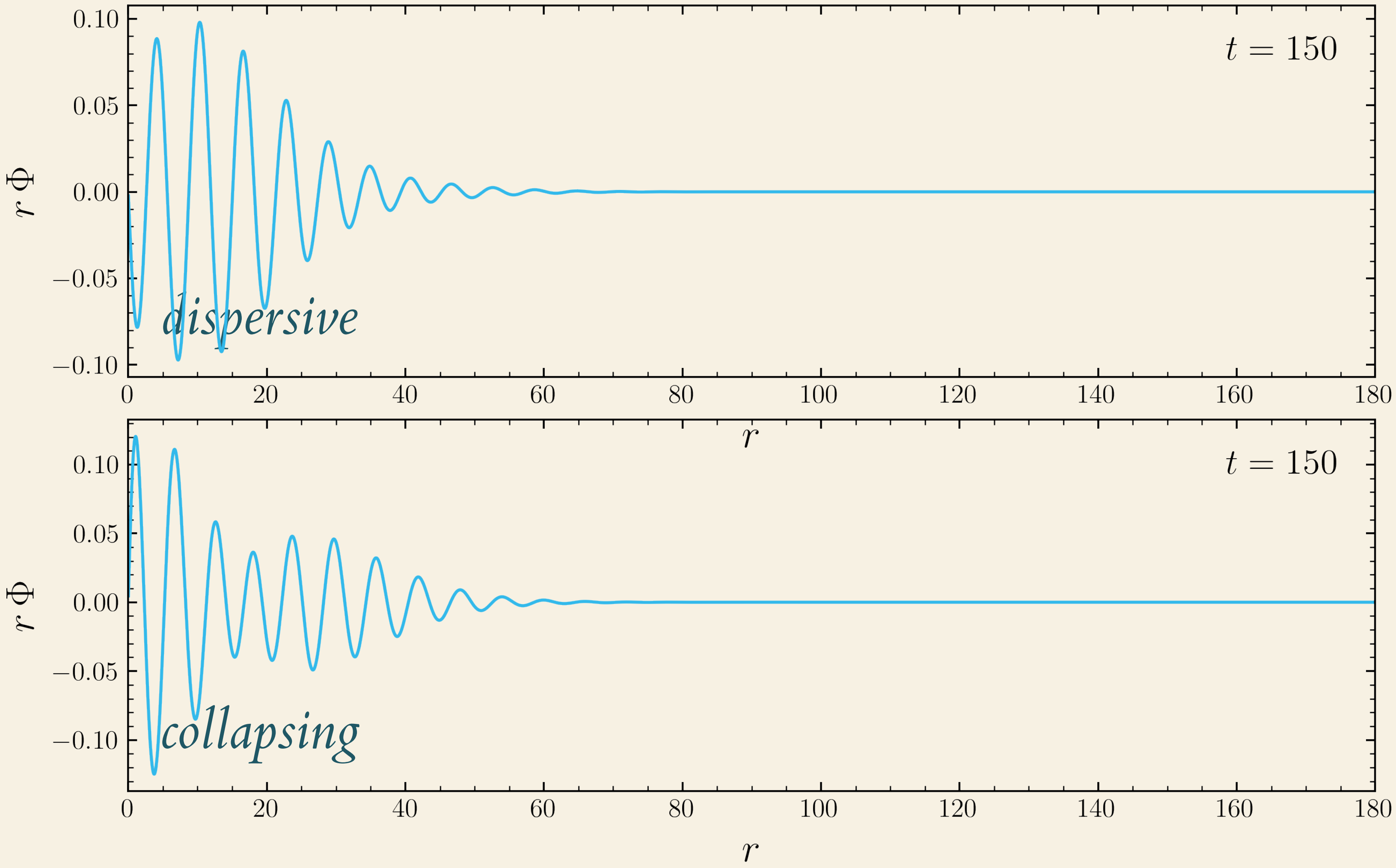

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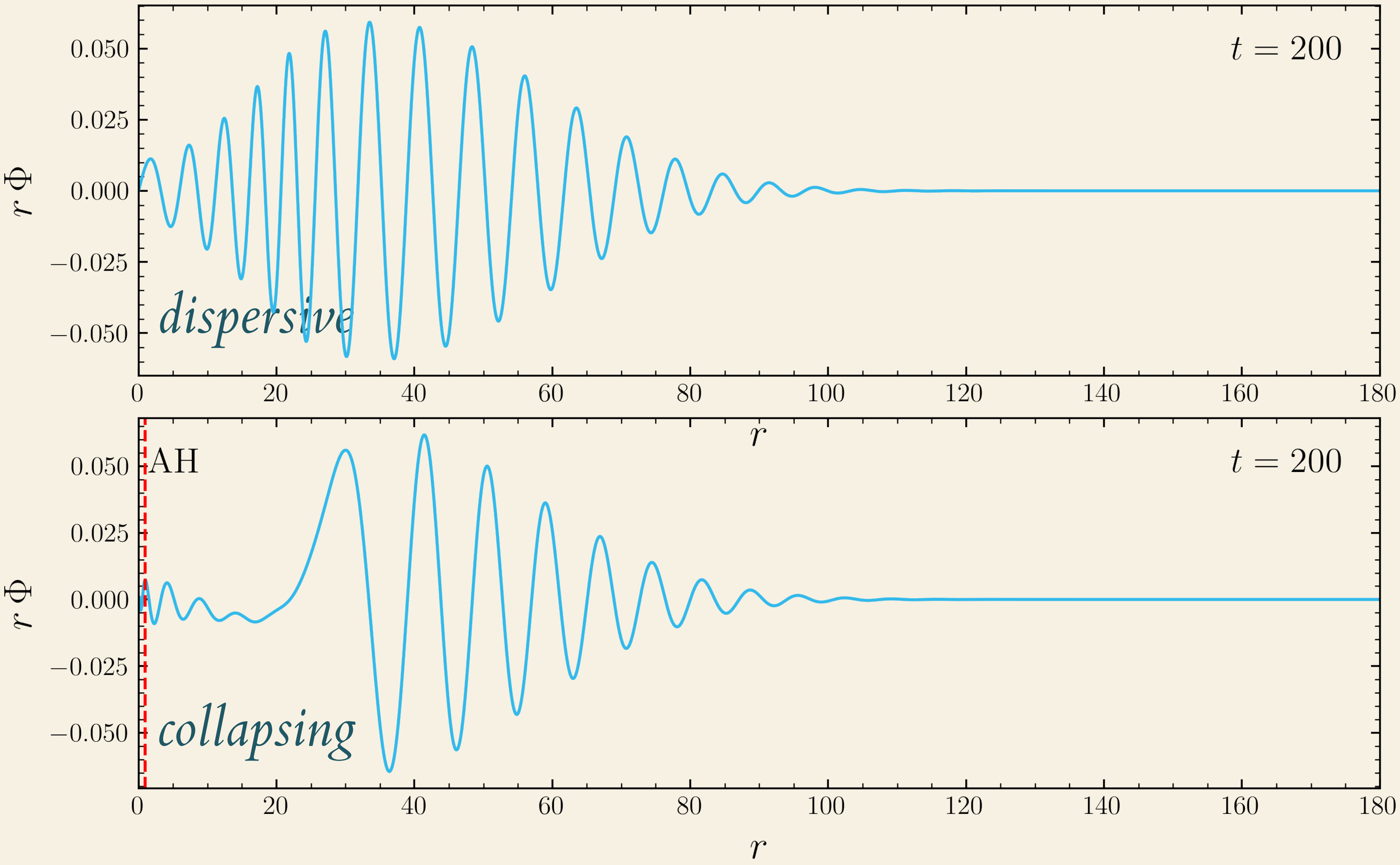

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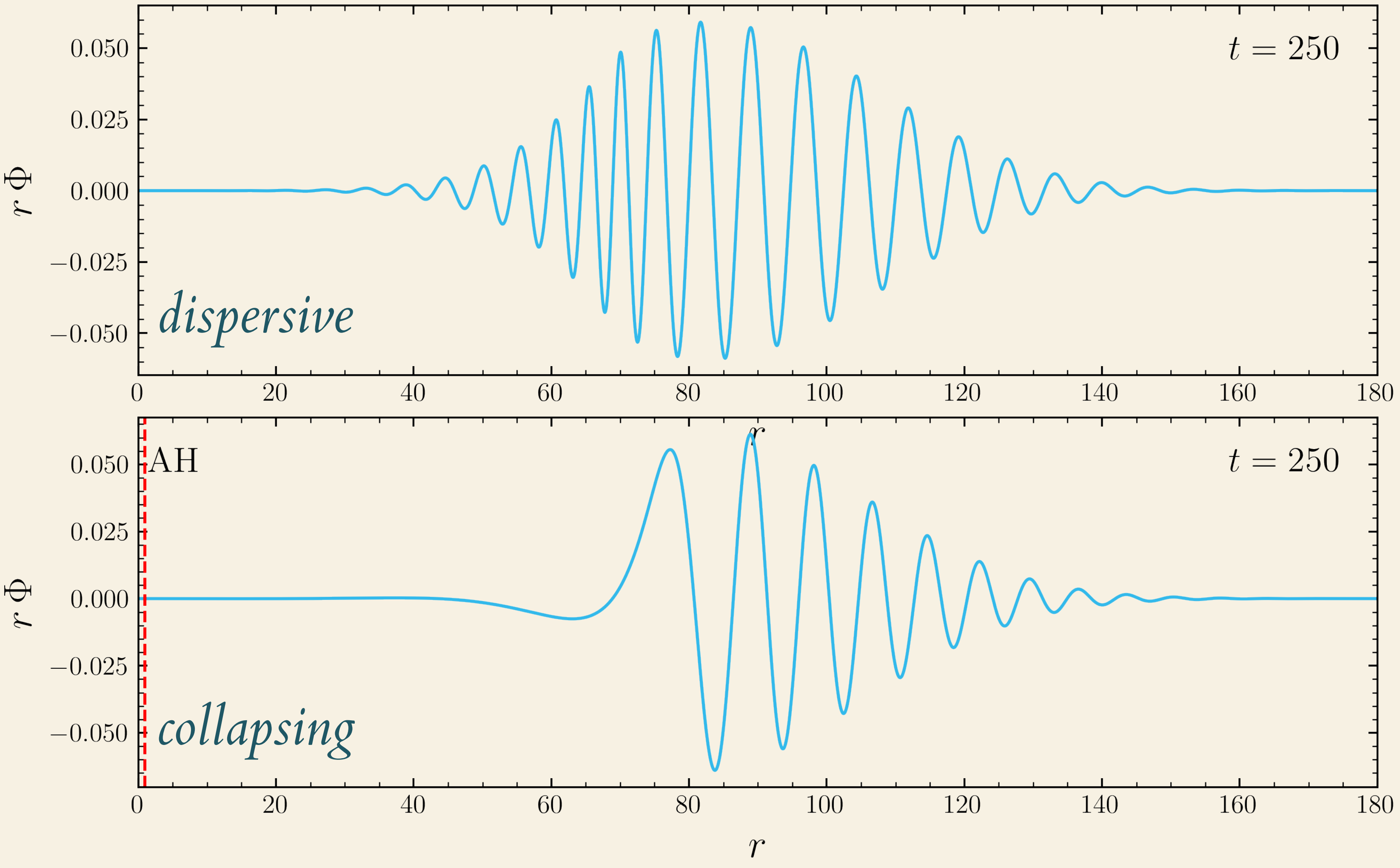

What can we expect?

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- A harmonic with frequency ω_0 whose amplitude scales as ε.
- A higher harmonic with frequency $3\omega_0$ whose amplitude scales as ε^3.
- A frequency shift, $\omega_0 \to \omega_0 + \delta\omega\varepsilon^2$.
- Faster decay of the higher harmonic, $A_{3\omega_0} \sim \mathcal{O}\left(r^{-2}\right)$.

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In the $\varepsilon > 1$ regime, and at late times (after BH formation):

$$\Phi \sim \sum A_n \exp(-i\omega_n t)$$

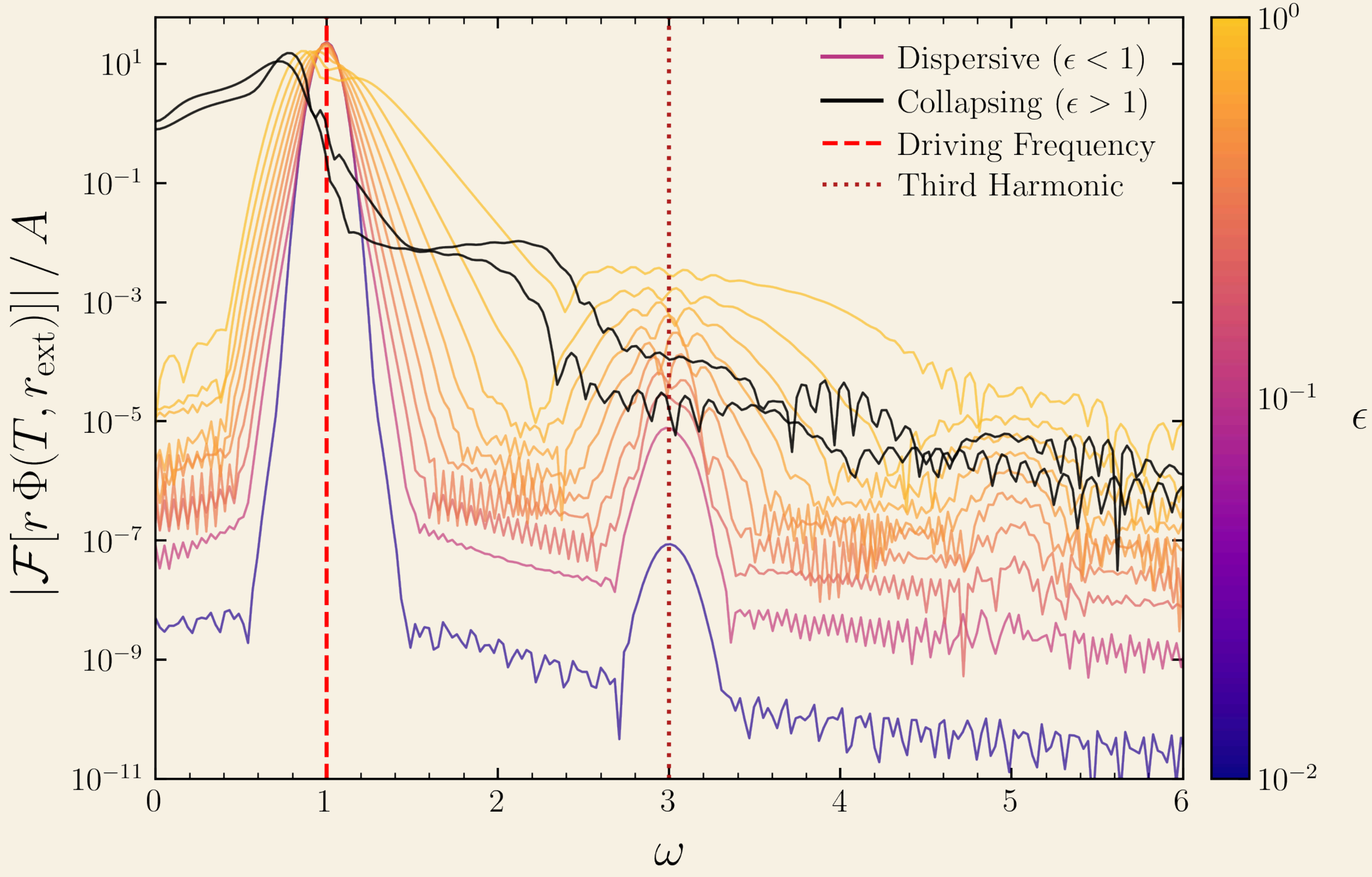
Evolution of the spectrum

Extract $r\Phi$ at some radius r_{ext}.

Fourier transform w.r.t. proper time.

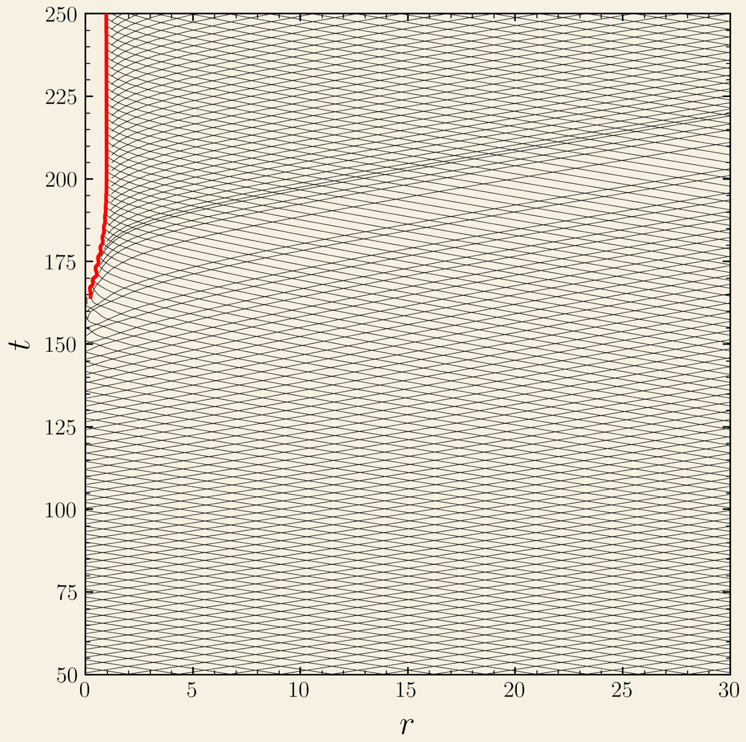
In the $\varepsilon \ll 1$ regime: quasi-discrete spectrum.

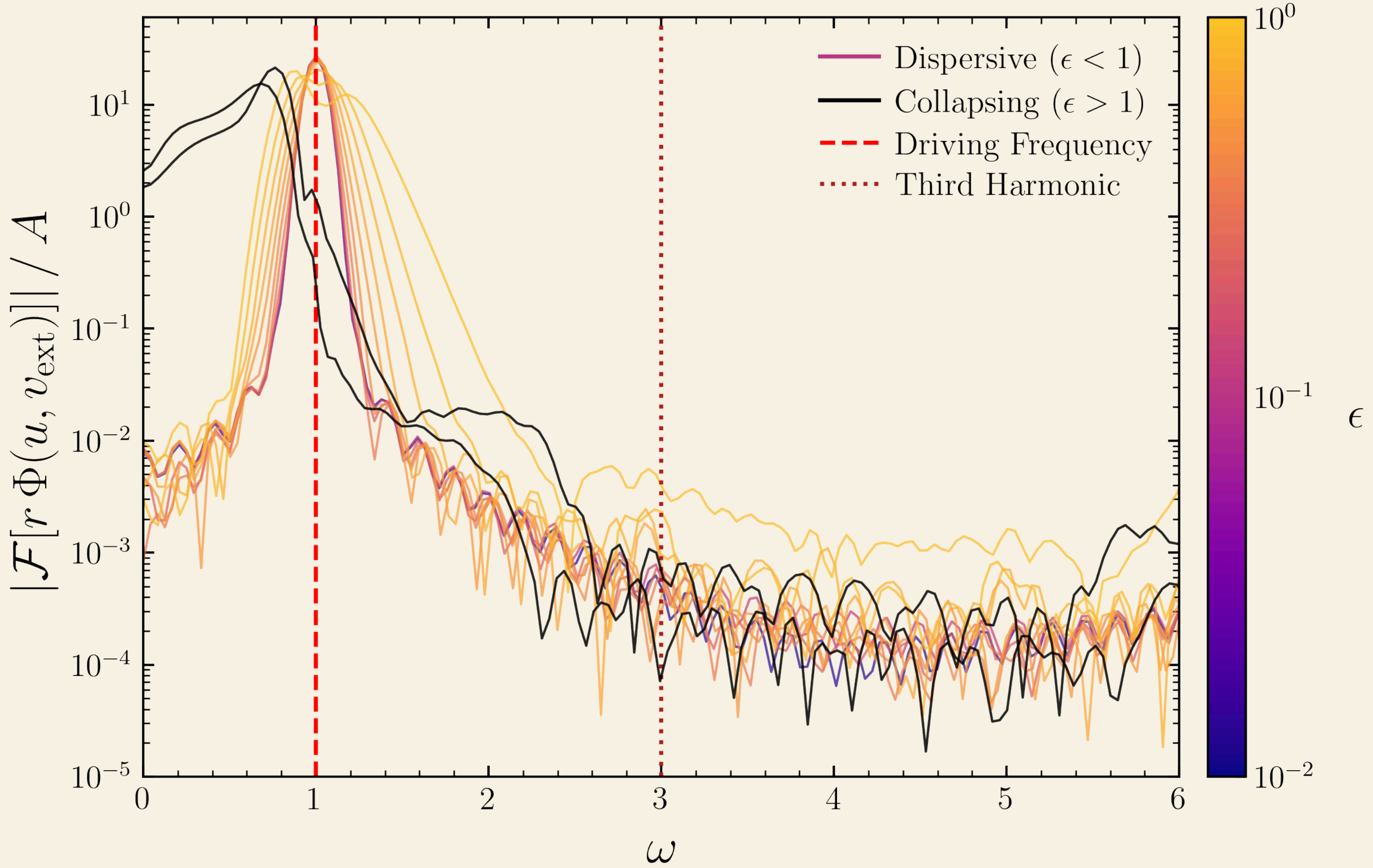
As $\varepsilon \to 1$, the spectrum flattens.



Evolution of the spectrum

Similar behavior along $\mathscr{I}^+$!





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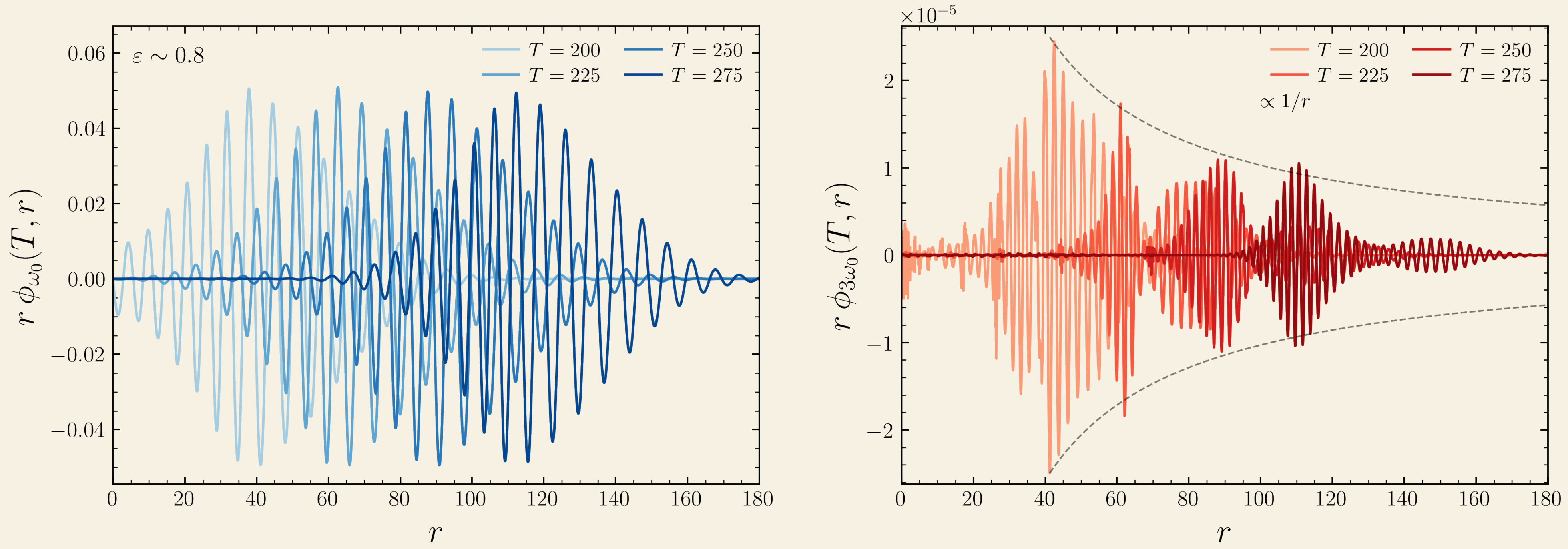
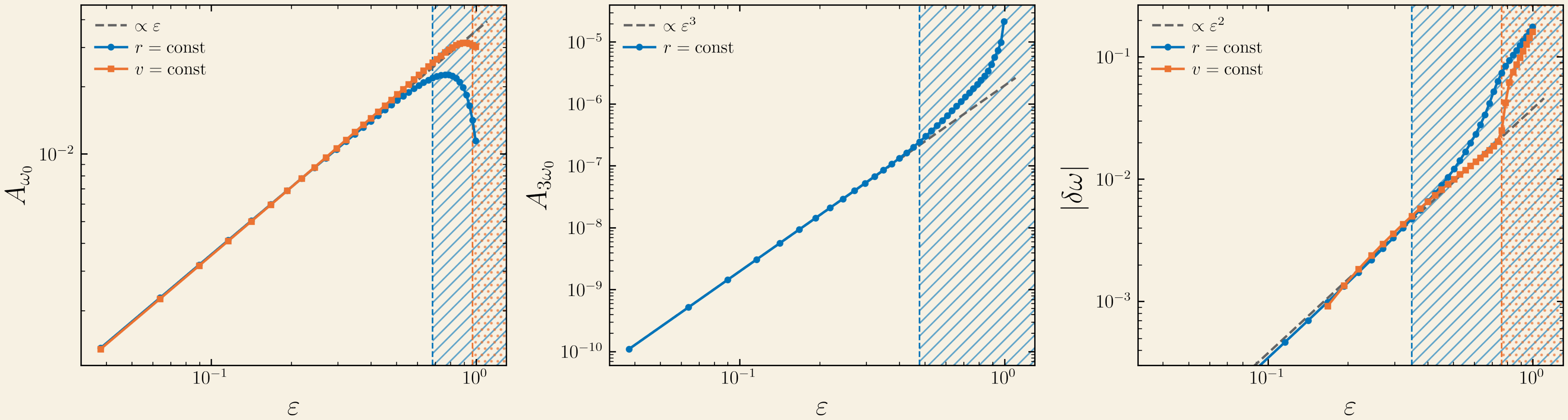
Higher harmonics don't reach $\mathscr{J}^+$

Perturbation theory predicts a higher harmonic, with frequency $3\omega_0$, that decays as $\mathcal{O}(r^{-2})$. We can isolate each frequency component and test this: $\varphi_\omega = \mathscr{F}^{-1}[W_\omega \mathscr{F}[\Phi]]$.



When does perturbation theory break?

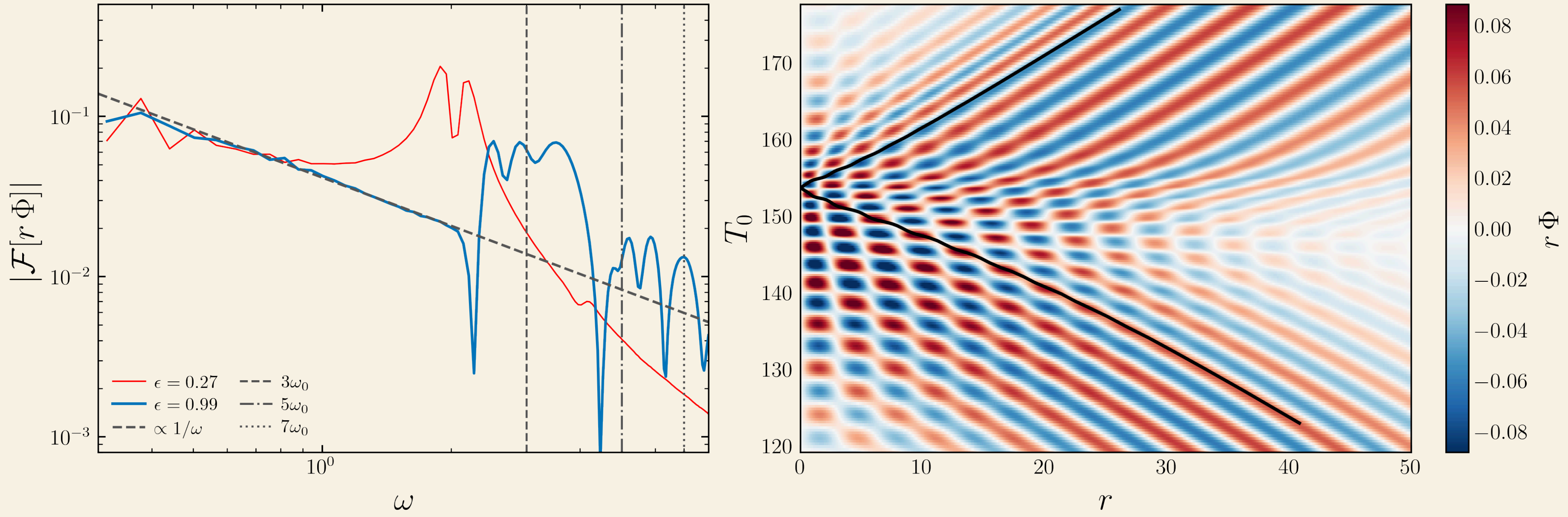
Quantify when the amplitudes deviate from perturbation theory by more than 10%,



Self-similarity near criticality

For $\varepsilon \approx 1$, discrete self-similarity emerges, $\Phi(\tau + 3.44) = \Phi(\tau)$, with $\tau \sim \log(T_* - T_0)$.
For the Fourier transform:

$$\int dT_0 \exp(i\omega T_0)\Phi(\tau) \sim \int dx \exp(i\omega x) x^{-2\pi\frac{i}{\Delta}} \sim \omega^{-1-2\pi\frac{i}{\Delta}}.$$



III Waves and black holes

in collaboration with


Vitor Cardoso

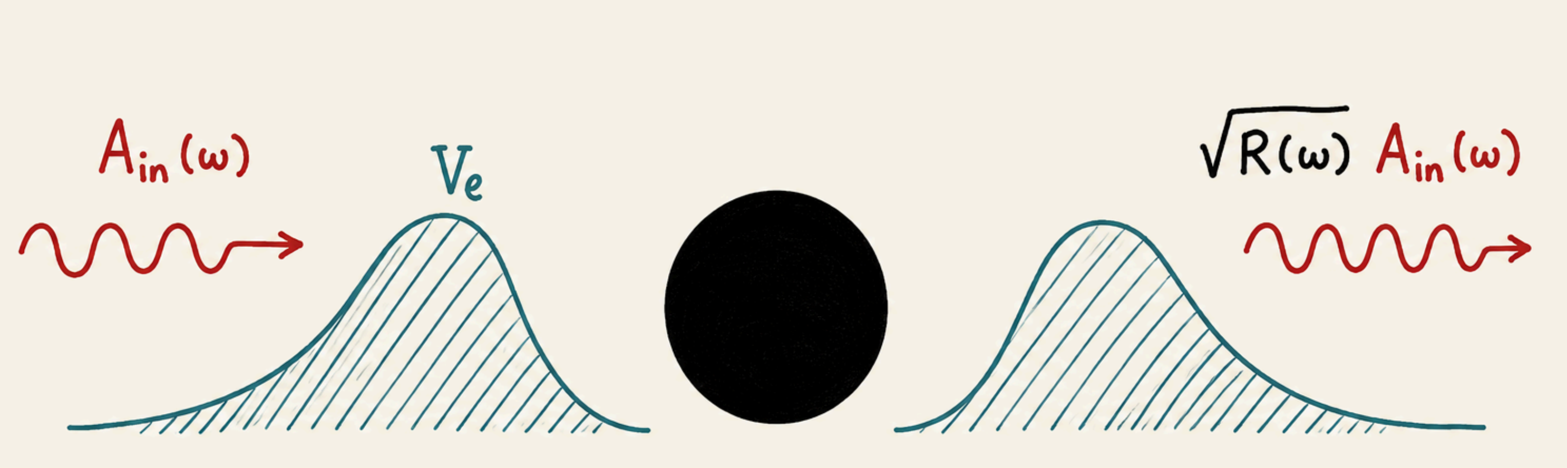

Furkan Tuncer


Ulrich Sperhake

How does a BH scatter GWs?

Let $g_{ab} = g^{Sch}_{ab} + \varepsilon h_{ab} \exp(-i\omega t) + \ldots$ describe a BH and an impinging monochromatic GW with frequency ω. What is the outgoing state?

To linear order: a spectrally filtered wave with ω (Compton)



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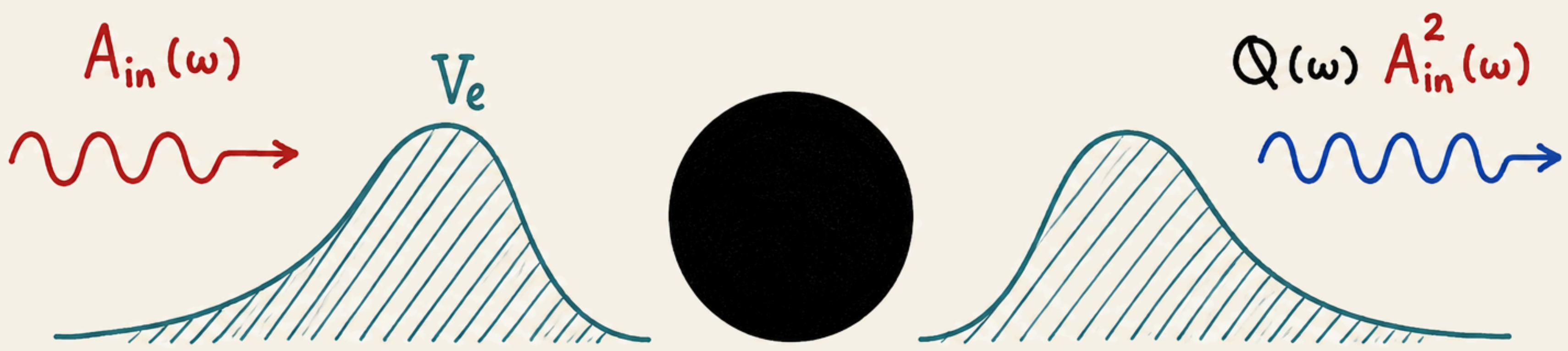
To second order: a higher harmonic, with frequency 2ω.



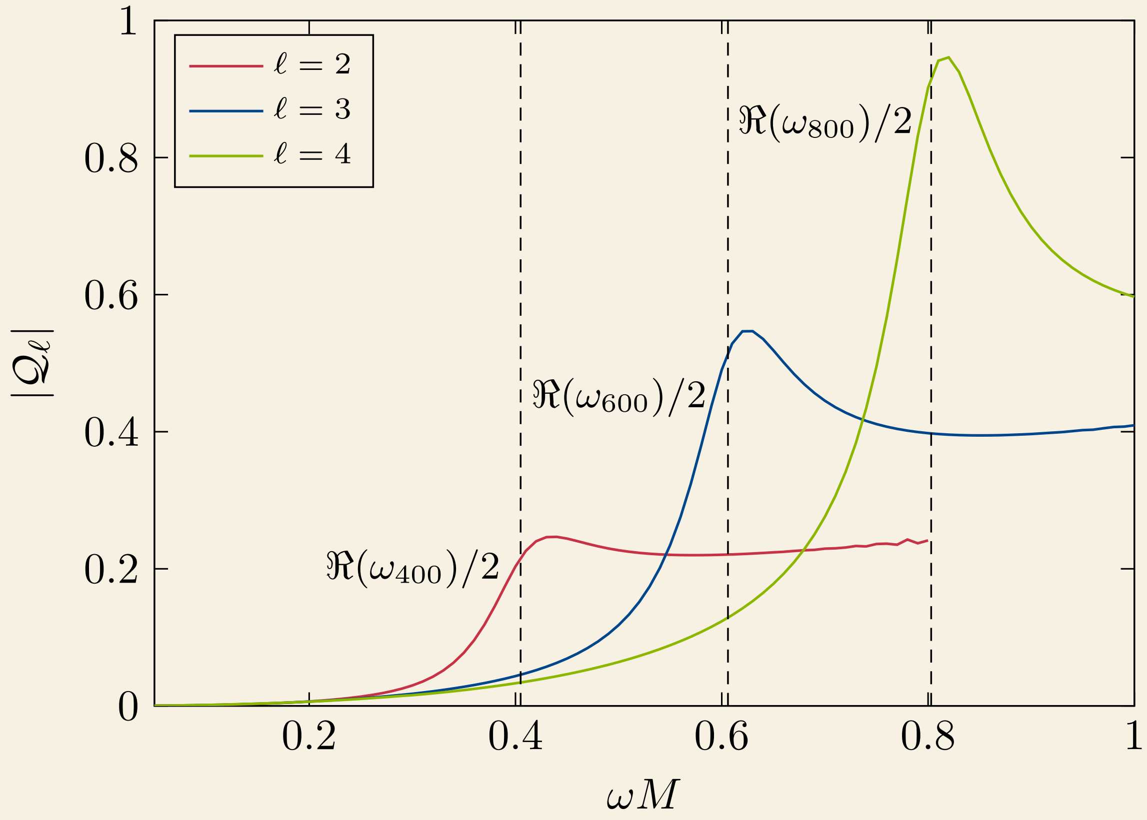
Higher harmonic excitation

Second-order perturbations around a Schwarzschild BH:

$$\frac{d^2\psi^{(2)}}{dr_*^2} + \left(4\omega^2 - V_{2\ell}\right)\psi^{(2)} = S\left(\psi^{(1)}, \psi^{(1)}\right).$$

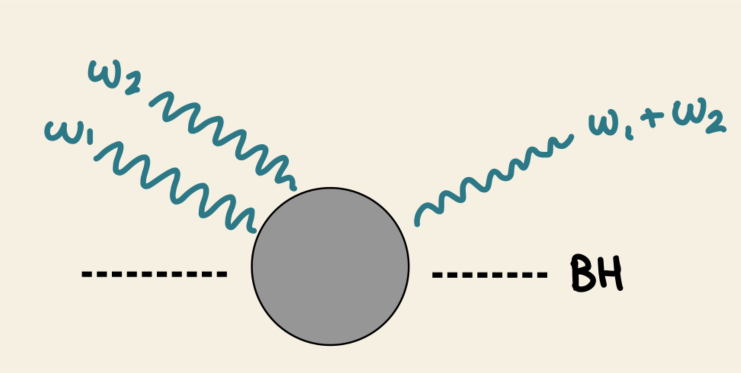
After regularising the source, asymptotically $h^{(2)} \sim Q(\omega) A_{in}^2 \exp(2i\omega r_*)$, with

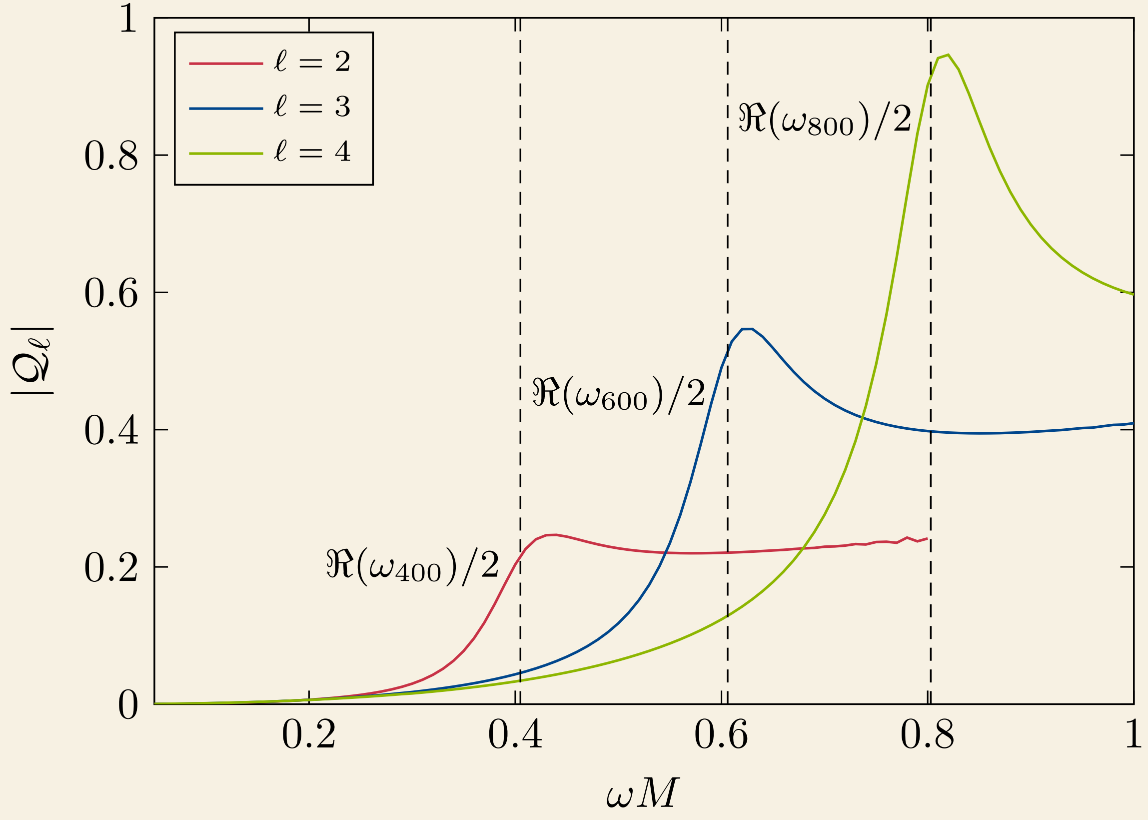
$$Q(\omega) = C(\ell)\frac{\int dr_* \mathfrak{S}\left(\psi^{(1)}\right)\psi_{in}}{4i\omega\mathcal{R}(\omega)^2}.$$



Higher harmonic excitation

- Resonances at QNM frequencies.
- Limits: vanishing as $\omega M \ll 1$ (kinematics), constant? as $\omega M \gg 1$
- Challenge: Matching from scattering amplitudes at $O(GM\omega)^2$.





Take-aways

What we showed

- ✓ Critical collapse as a set-up to study convergence of perturbation theory.
- ✓ (Linear) perturbation theory is not sufficient.
- ✓ Non-perturbative effects visible at $\varepsilon < 1$.
- ✓ Nonlinear effects in BHs are small at $\omega < \omega_{QNM}$ (inspiral-driven, unless $\gamma \gg 1$).
- ✓ Resonant excitation of higher harmonics when driving a BH at QNM frequency (triples?).

Open questions

- → *Can resummations of perturbation theory explain critical collapse?*
- → *Scattering amplitudes: $4 + k$-point amplitude $= (k+1)$th BHPT?*
- → *Nonlinear scattering of waves off rapidly rotating Kerr BHs?*

ありがとうございます