# Strong Field GW Lensing

*Lensing in Triple Systems*

Jaime Redondo-Yuste

*RESCEU Seminar · Tokyo · 2026*

*Based on 2502.14073 & 2605.20320*

JOHNS HOPKINS UNIVERSITY

YITP

PRINCETON UNIVERSITY

天国と地獄 (1963) - 黒沢 明

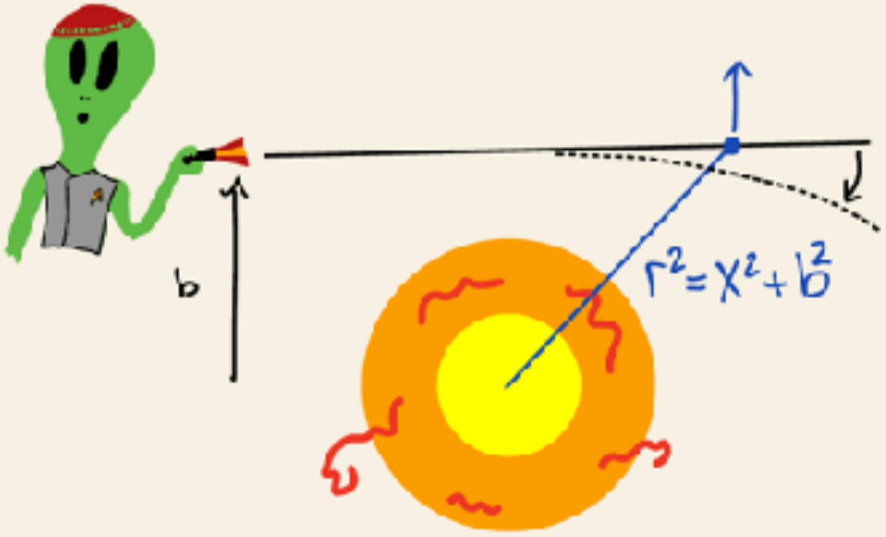
# Outline

# I Lensing of Gravitational Waves

# Gravity Bends Light



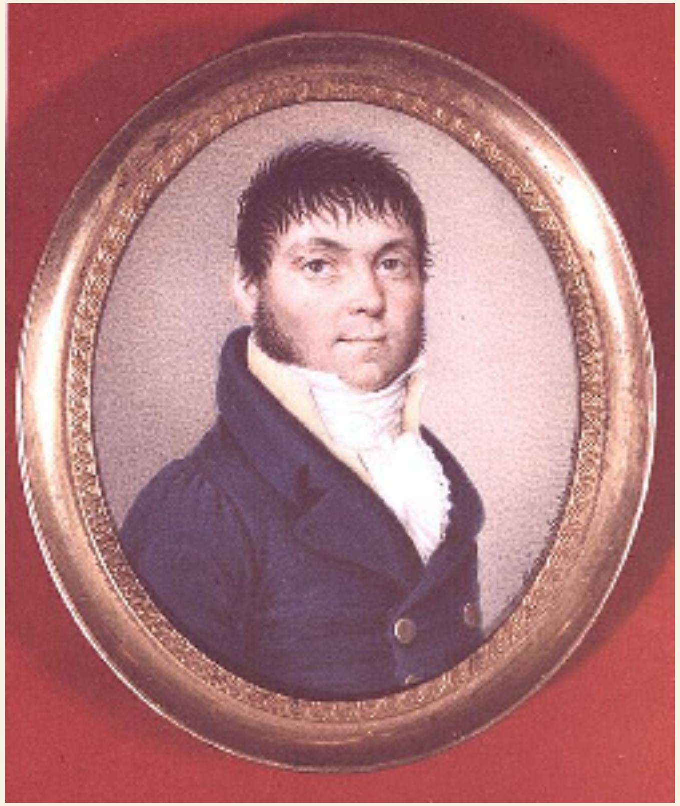
Newtonian gravity predicts gravitational lensing




$$F_\perp = \frac{GMmb}{r^3}$$

$$\delta = \frac{\Delta p_\perp}{p} = \frac{\int F_\perp \, dt}{mc}$$

$$= \frac{GMb}{c^2} \int \frac{dx}{(x^2+b^2)^{3/2}} = \underline{\underline{\frac{2GM}{c^2 b}}} \, .$$



Johan Georg von Soldner



*[Soldner (1801)]*

# Einstein's Gravity Bends Light More



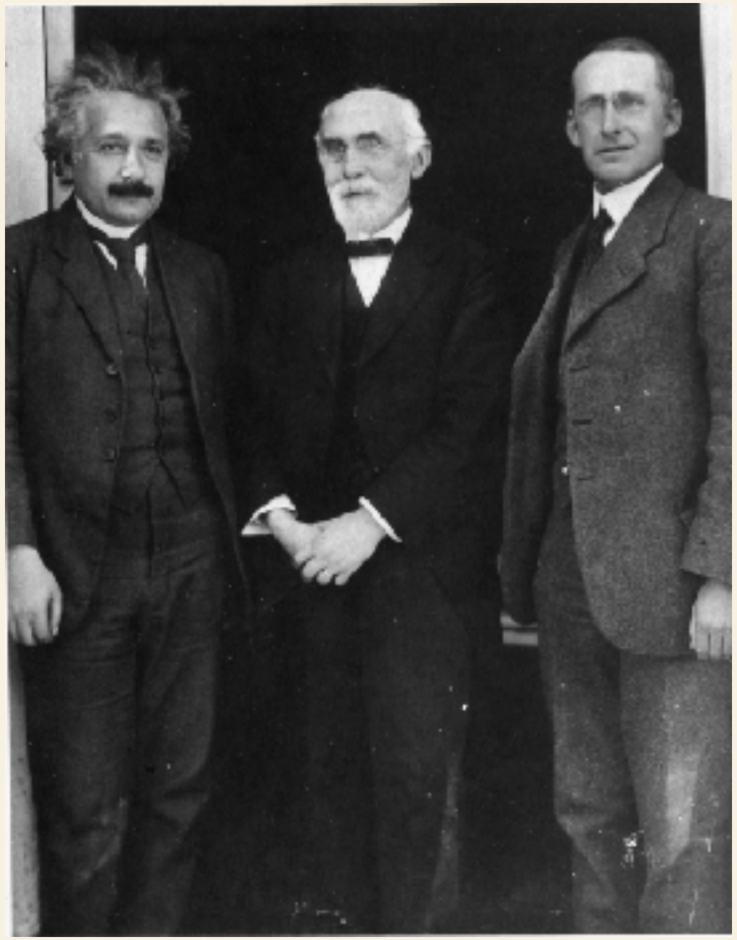
For a star on the limb of the sun: $\delta_N = 0.83\text{arcsec}$

Einstein (1911): $\delta_{GR} = 0.83\text{arcsec}$ [wrong!]

1912,14: Failed expeditions to Brazil (rain) & Crimea (war)

Einstein (1915): $\delta_{GR} = 2\delta_N = 1.75\text{arcsec}$ [correct!]

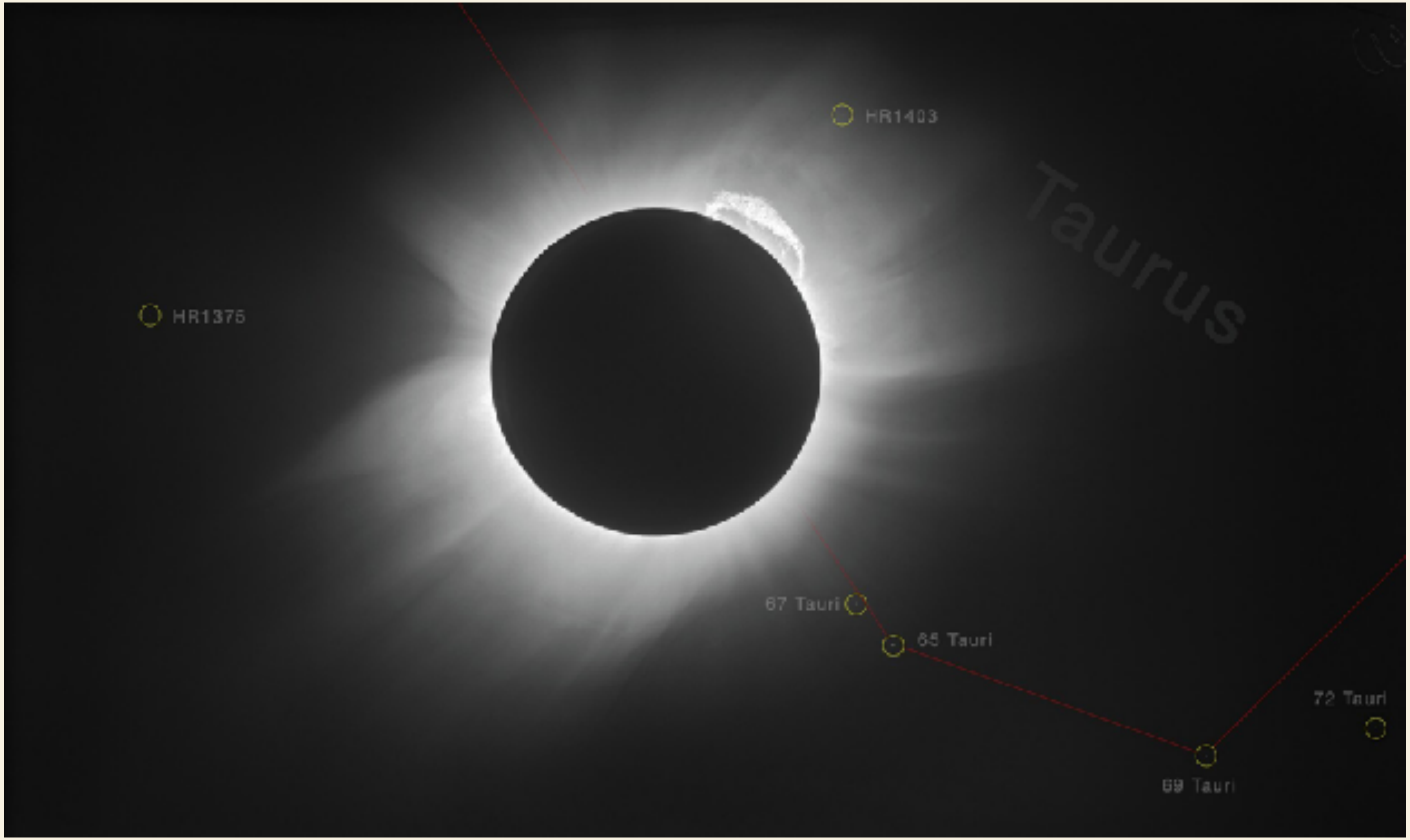
1919: Succesful expedition by Dyson & Eddington



Einstein, De Sitter, Eddington

# Einstein's Gravity Bends Light More







LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

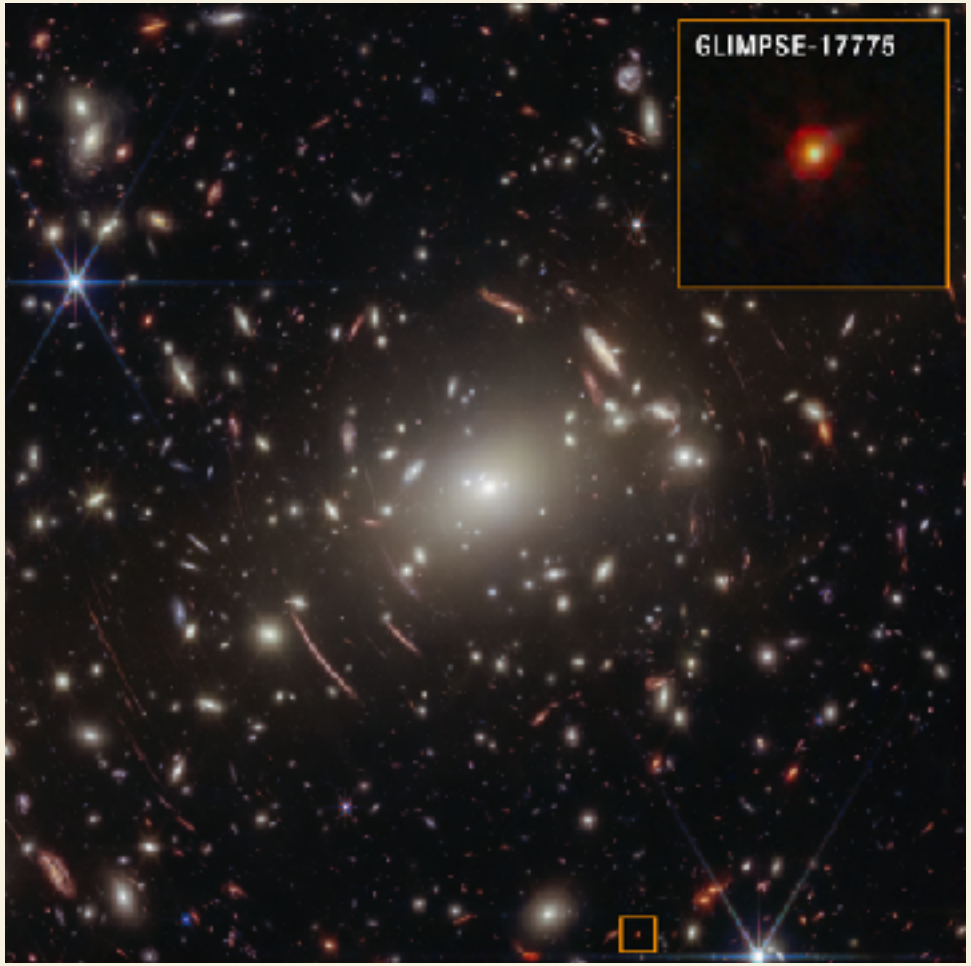
EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

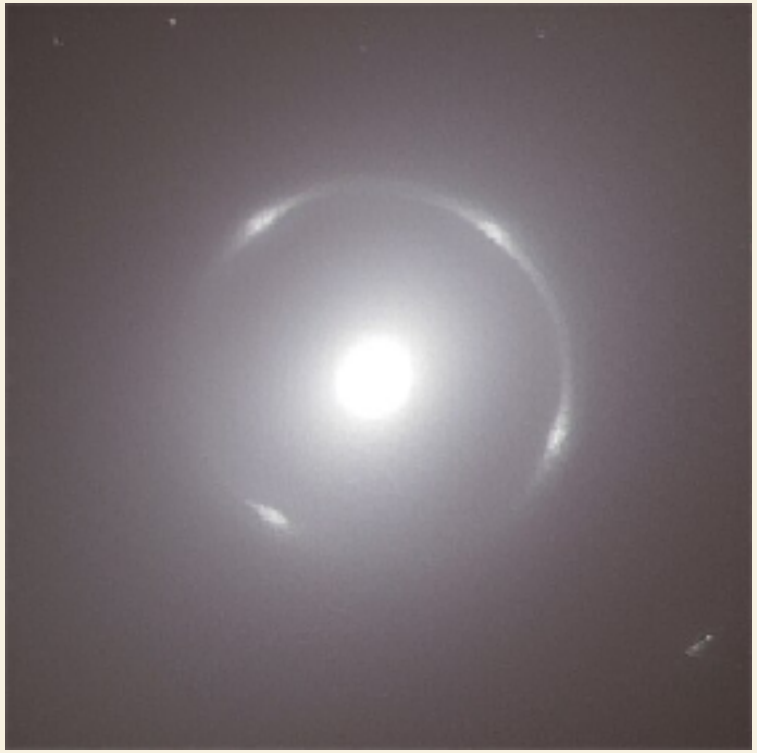
No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

# Gravitational Lensing Today




*[NASA, ESA, CSA, Vasily Kokorev (UT Austin); Image Processing: Alyssa Pagan (STScI)]*

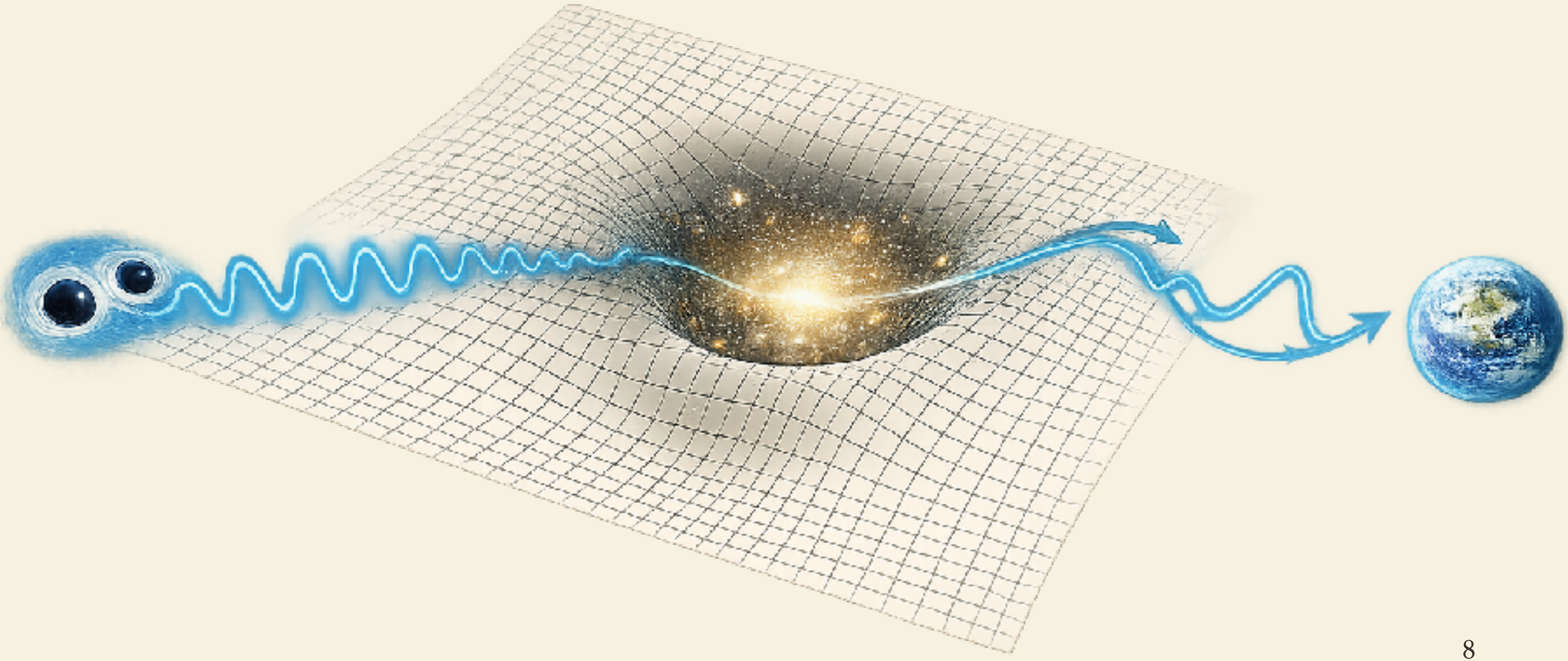
Little Red Dot in Abell-S1063 at redshift 3.5, only visible because it's magnified via lensing!



Einstein Ring: NGC6505 lensing a more distant galaxy,

*[ESA/Euclid/Euclid Consortium/NASA, image processing by J.-C. Cuillandre, G. Anselmi, T. Li]*

# Gravitational Waves are also Lensed



# Gravitational Waves are also Lensed



Maxwell equations in curved space: $\Box A^\mu - R^\mu_\nu A^\nu = 0$

(Linearized) Einstein equations in curved space: $\Box \bar{h}_{\mu\nu} + 2R_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} = 0$

Eikonal ansatz: $X_{\mu\ldots} = \mathscr{A}\epsilon_{\mu\ldots}e^{iS/\epsilon}$, with $\epsilon = \dfrac{[\text{wavelength}]}{[\text{curv . scale}]} \ll 1$

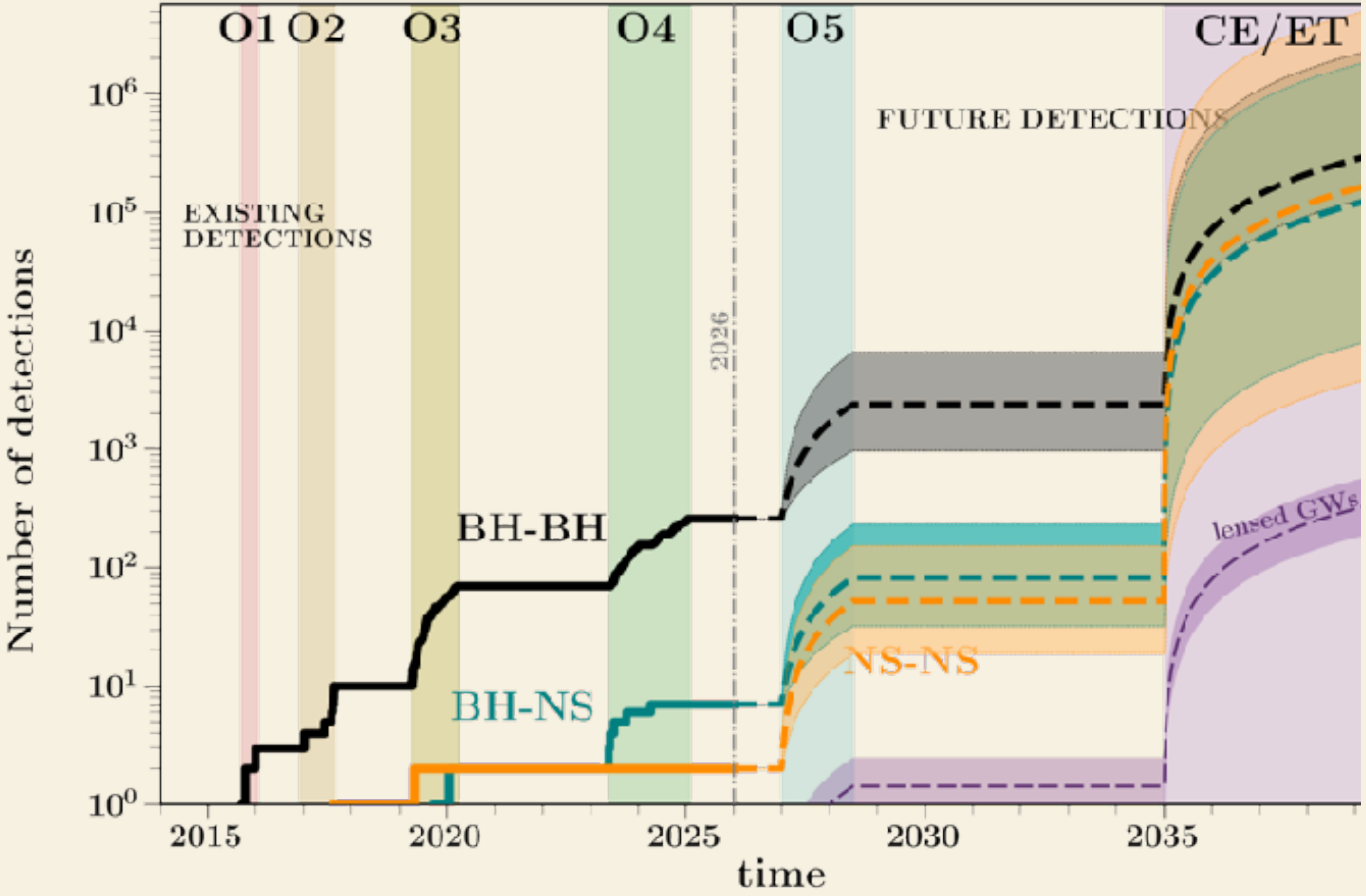
At $O(\epsilon^{-2})$: $k_\mu k^\mu = k^\mu \nabla_\mu k^\nu = 0$, with $k_\mu = \partial_\mu S$, [null geodesic]

At $O(\epsilon^{-1})$: $k^\mu \nabla_\mu \epsilon_{\alpha\beta} = k^\mu \epsilon_{\mu\alpha} = 0$, [transverse + parallel-propagation]

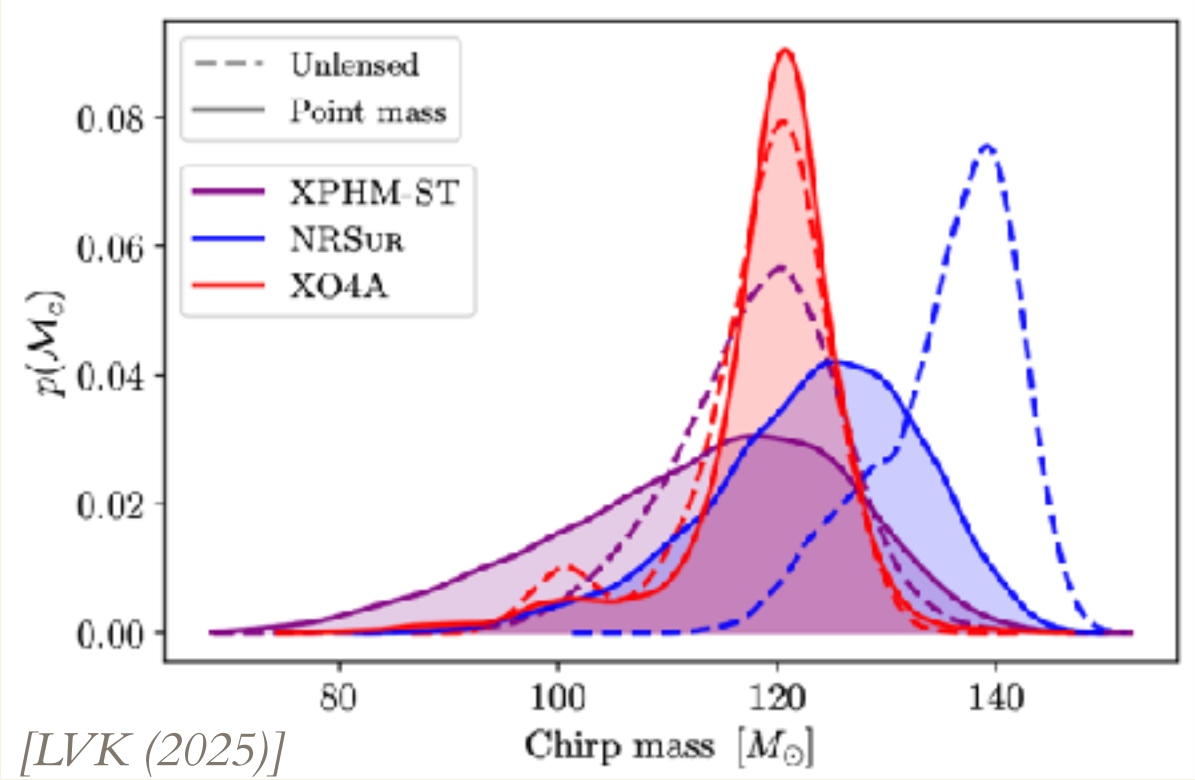
At $O(\epsilon^0)$: $\Box(\mathscr{A}\epsilon_{\mu\nu}) = 0$, & spin-curvature coupling…

# The Detection of the First Lensed GW is nigh!

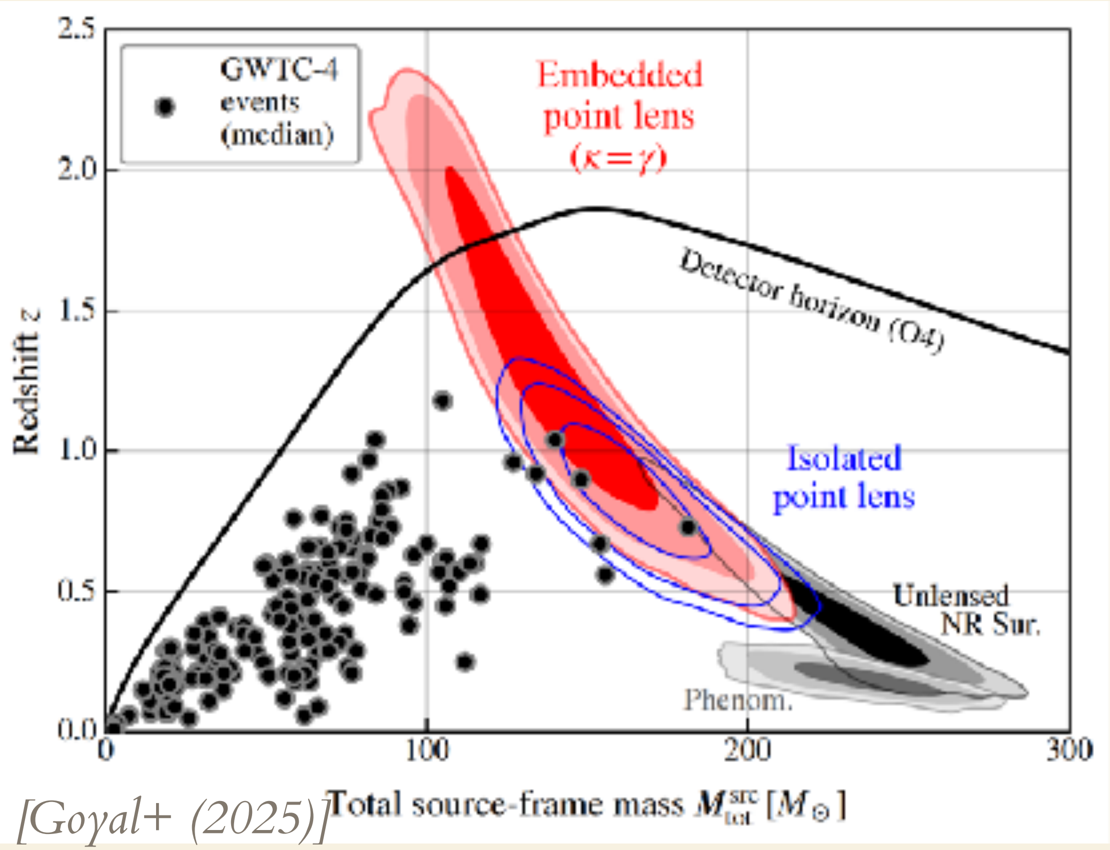




# A Case Study: GW231123



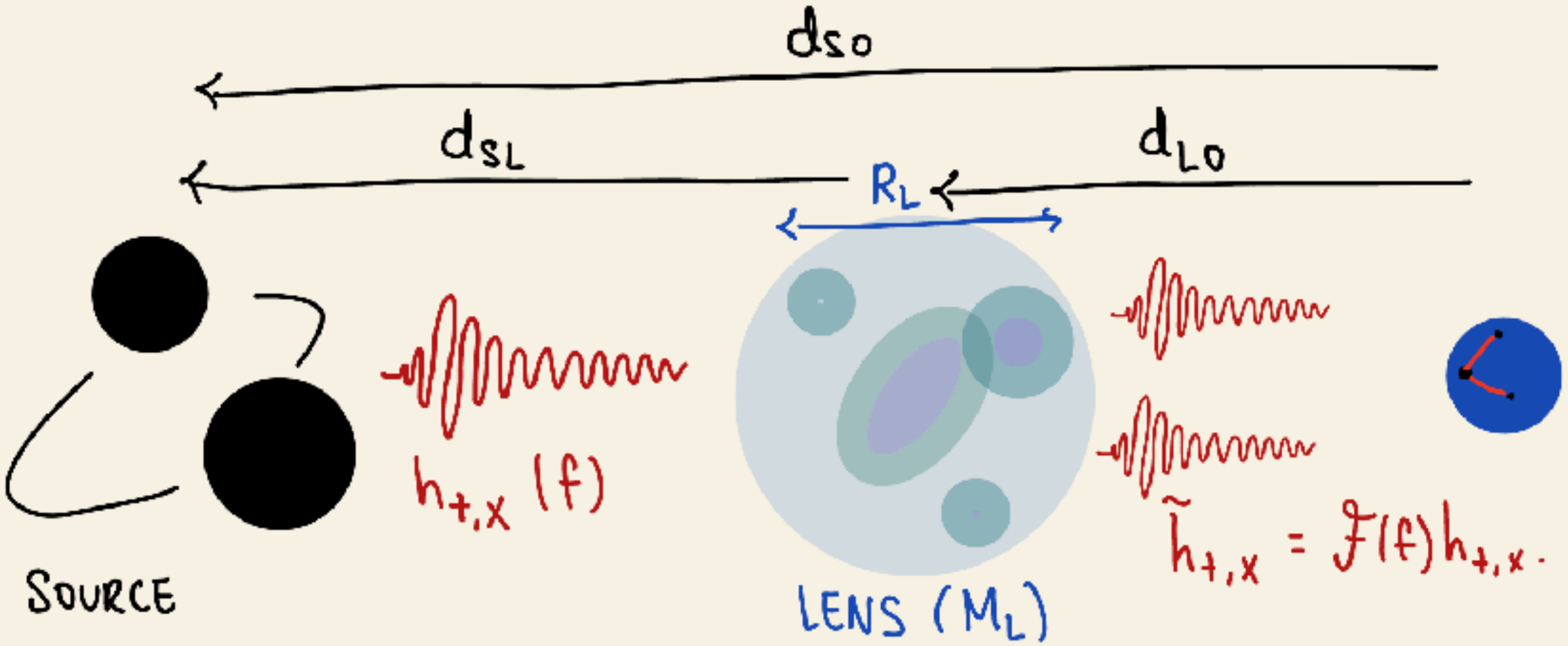

*[LVK (2025)]*




*[Goyal+ (2025)]*

“The probability of an astrophysical background of unlensed signals producing this level of consistency with a lensed waveform is < 0.39% (< 28%) without (with) the inclusion of the trials factor, where the significance is limited by the size of our background simulations.” [LVK]

# Lensing 101



# Lensing 101



Weak field: $ds^2 = a(\eta)^2\left[-(1+2\Phi)d\eta^2 + (1-2\Phi)d\vec{x}^2\right]$, with $\Phi \sim M_L/R_L \ll 1$

GW Propagation: $\Box(h_{+,\times}\epsilon_{\mu\nu}^{+,\times}) \approx \Box h_{+,\times} = 0$, [neglect polarization effects]

In the frequency domain: $(\nabla^2 + \omega^2)h = 4\Phi\omega^2 h$

Thin lens ($R_L \ll D_L, D_{SL}$): $F(w,y) = \dfrac{w}{2\pi i}\displaystyle\int d^2x \exp[iwT_d(x,y)]$

# Lensing 101



Thin lens ($R_L \ll D_L, D_{SL}$): $\quad F(w,y) = \frac{w}{2\pi i}\int d^2x \exp[iwT_d(x,y)]$

If $w \gg 1$: geometric optics, and integral is dominated by saddle points (images)

time delay

$$F(w,y) \approx \sum_i \sqrt{\mu_i} \exp(i\omega t_j - i\pi n_j)$$

magnification

phase shift

If $w \lesssim 1$: wave optics. For simple lenses, analytical solutions. Otherwise: numerics*

$$F(w,y) = \exp\left(\frac{\pi w}{4} + \frac{iw}{2}[\log(w/2) - 2\phi(y)]\right)\Gamma(1 - iw/2)\,{}_1F_1(iw/2, 1; iwy^2/2)$$

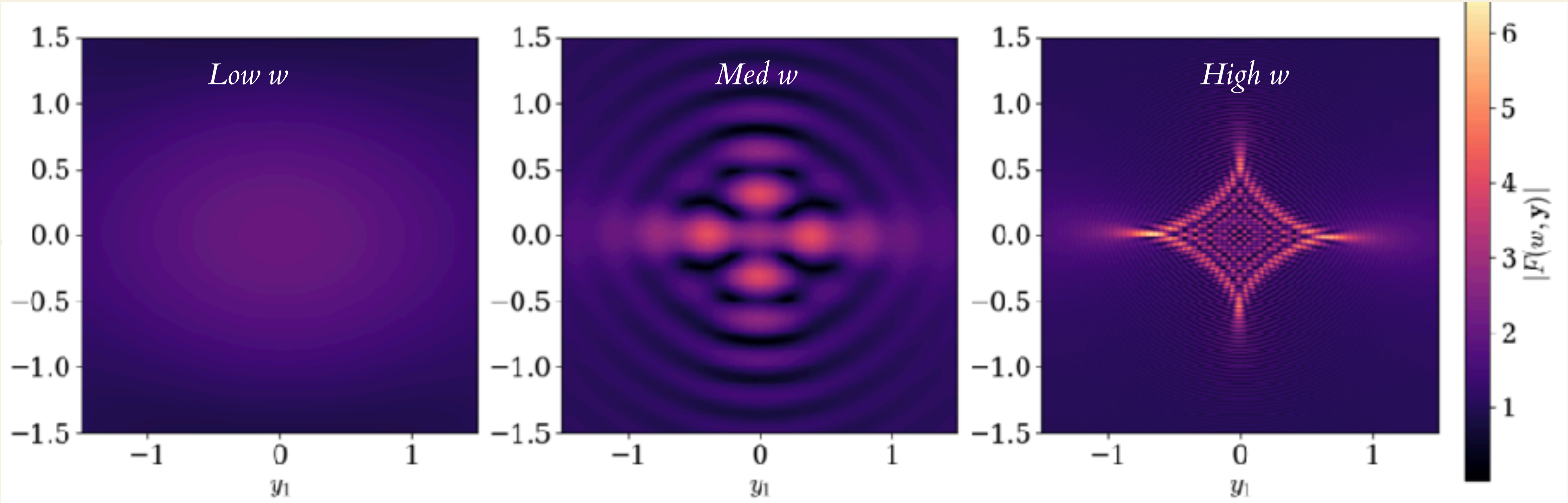


*[Takahashi & Nakamura (2003)]*

*[Villarrubia-Rojo+ (2025), Ephremizde (2026)]*

# Lensing 101

Thin lens ($R_L \ll D_L, D_{SL}$): $\quad F(w,y) = \dfrac{w}{2\pi i}\displaystyle\int d^2x \exp[iwT_d(x,y)]$




[Ephremizde (2026)]

# II GWs lensed by a Black Hole

*[based on 2502.14073]*
*in collaboration with*


Juno Chan
(NBI)

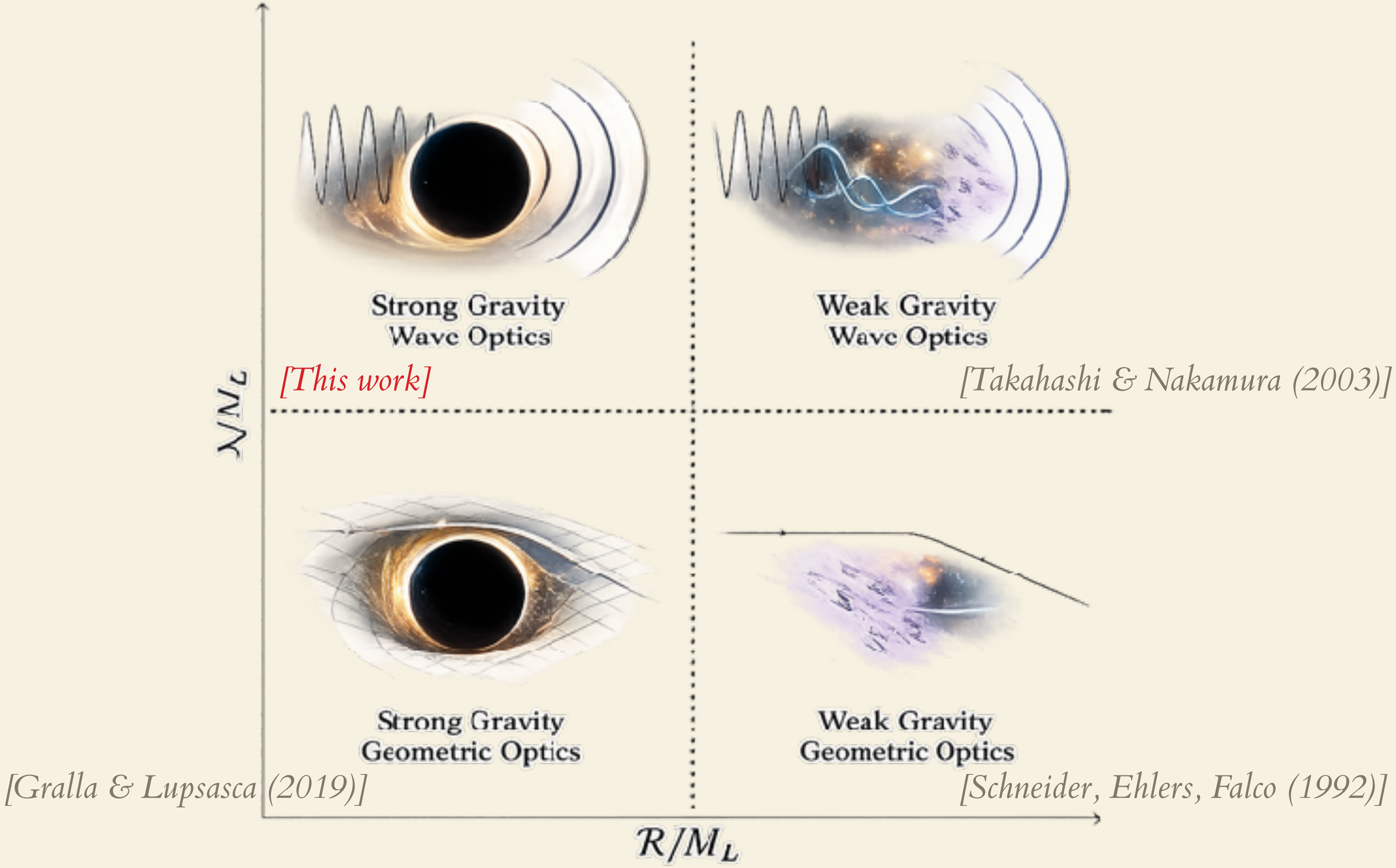
Conor Dyson
(>Birmingham)

Matilde Garcia
(Milano-Bicocca)

Luka Vujeva
(NBI)

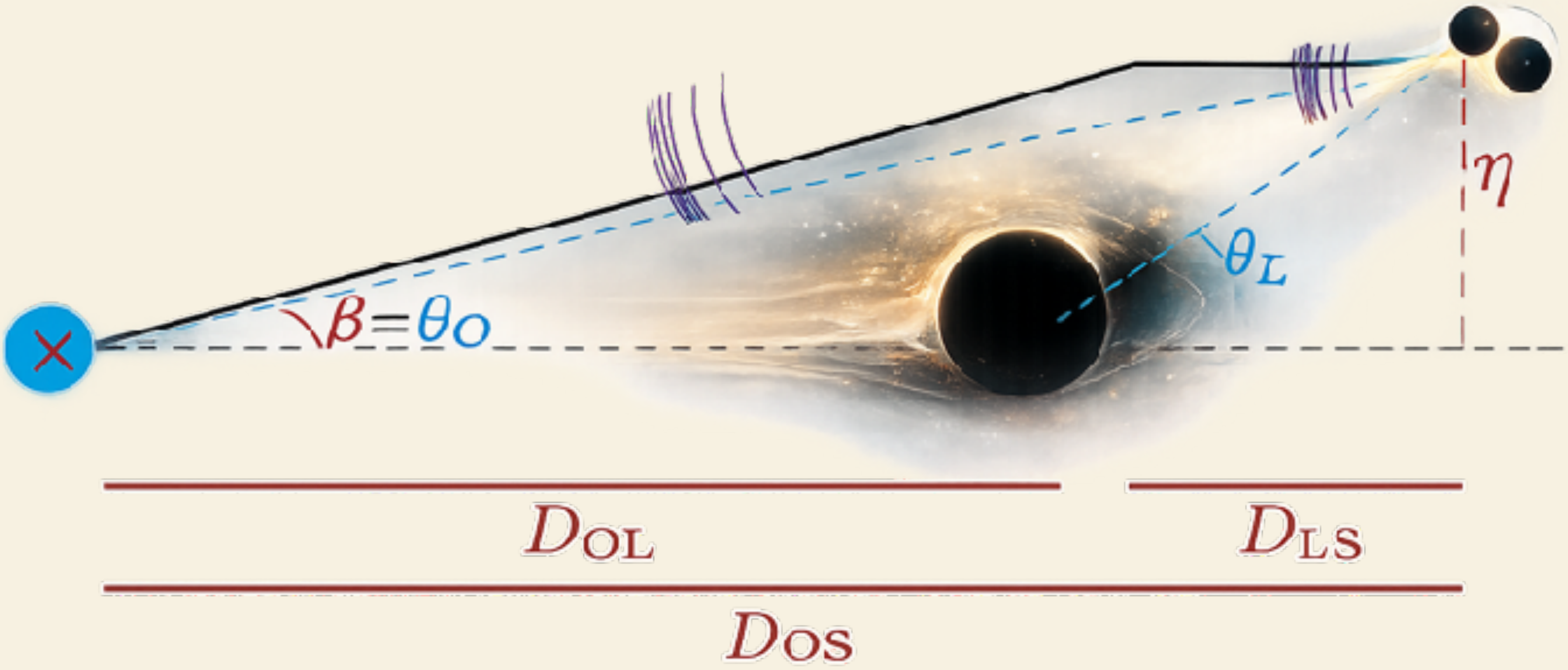
# Project: Lenshouse

# Regimes of GW Lensing



# Set-Up






Only assumptions: $D_{\mathrm{OL}} \sim D_{\mathrm{OS}} \gg D_{\mathrm{LS}} \gg \lambda$

# Lensing | BHPT



Dictionary between Lensing and BH Perturbation Theory

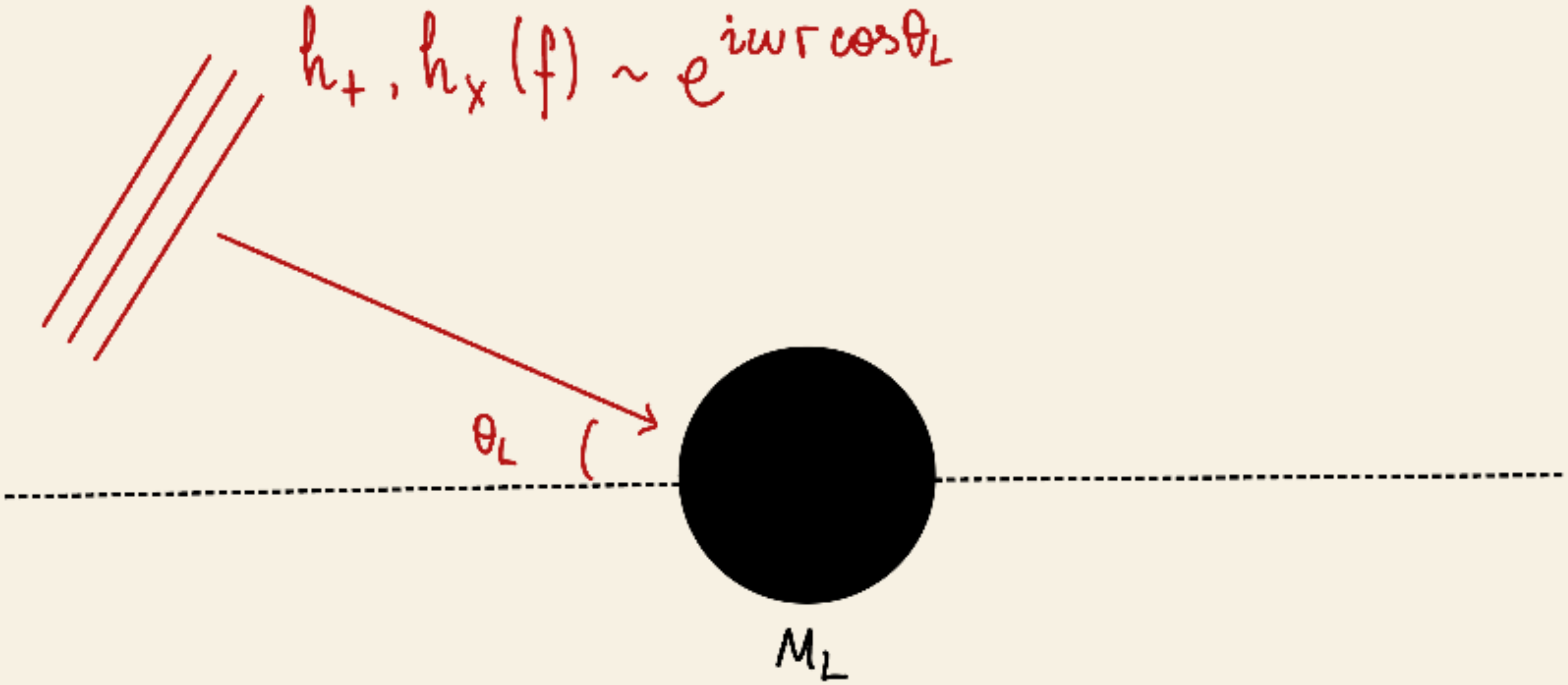
**Lensing**

- Plus/cross polarization in TT gauge
- Plane Waves $h \sim e^{i\omega z}$
- Dimensionless frequency $\varpi$
- Impact parameter $y$

**BHPT**

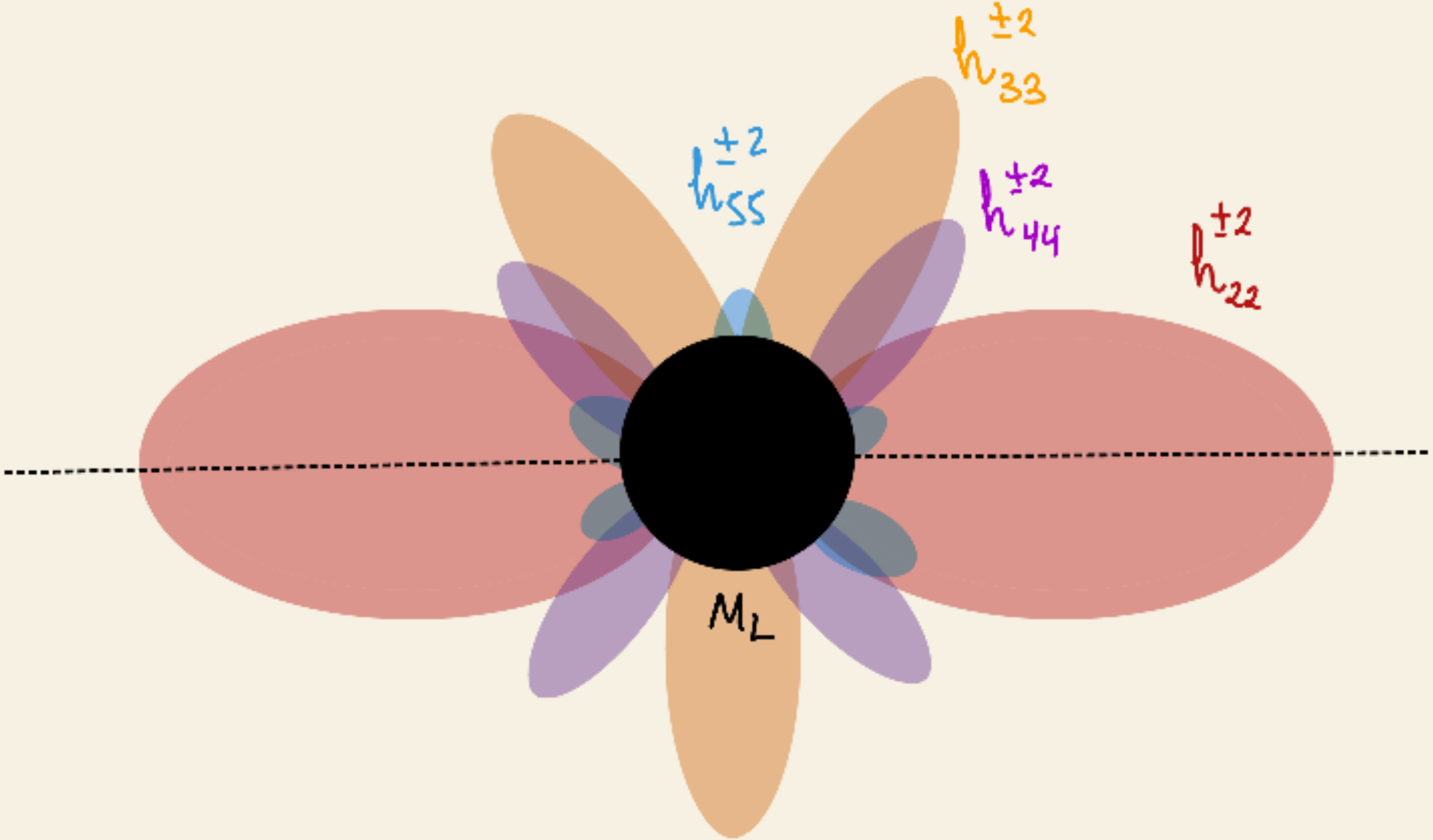
- Helicity eigenstates in RW gauge
- Partial Waves $h \sim \sum_{\ell m} e^{i\omega r} Y_{\ell m}$
- Frequency $\omega M_L = \dfrac{\varpi}{4(1+z)}$
- Incidence angle $\tan \theta_L = \dfrac{\theta_E y D_{\rm OL}}{D_{\rm LS}}$

# Lensing | BHPT



Lensing | BHPT





# Scattering of GWs by a BH



Master function $\psi_{\ell mp}$ constructed from $h^{\pm 2}_{\ell m}$ in RW gauge [need TT > RW at $r/M \gg 1$]

$$\frac{d^2\psi_{\ell mp}}{dr_*^2} + (\omega^2 - V_{\ell p})\psi_{\ell mp} = 0$$

In-/Out-going amplitudes are related

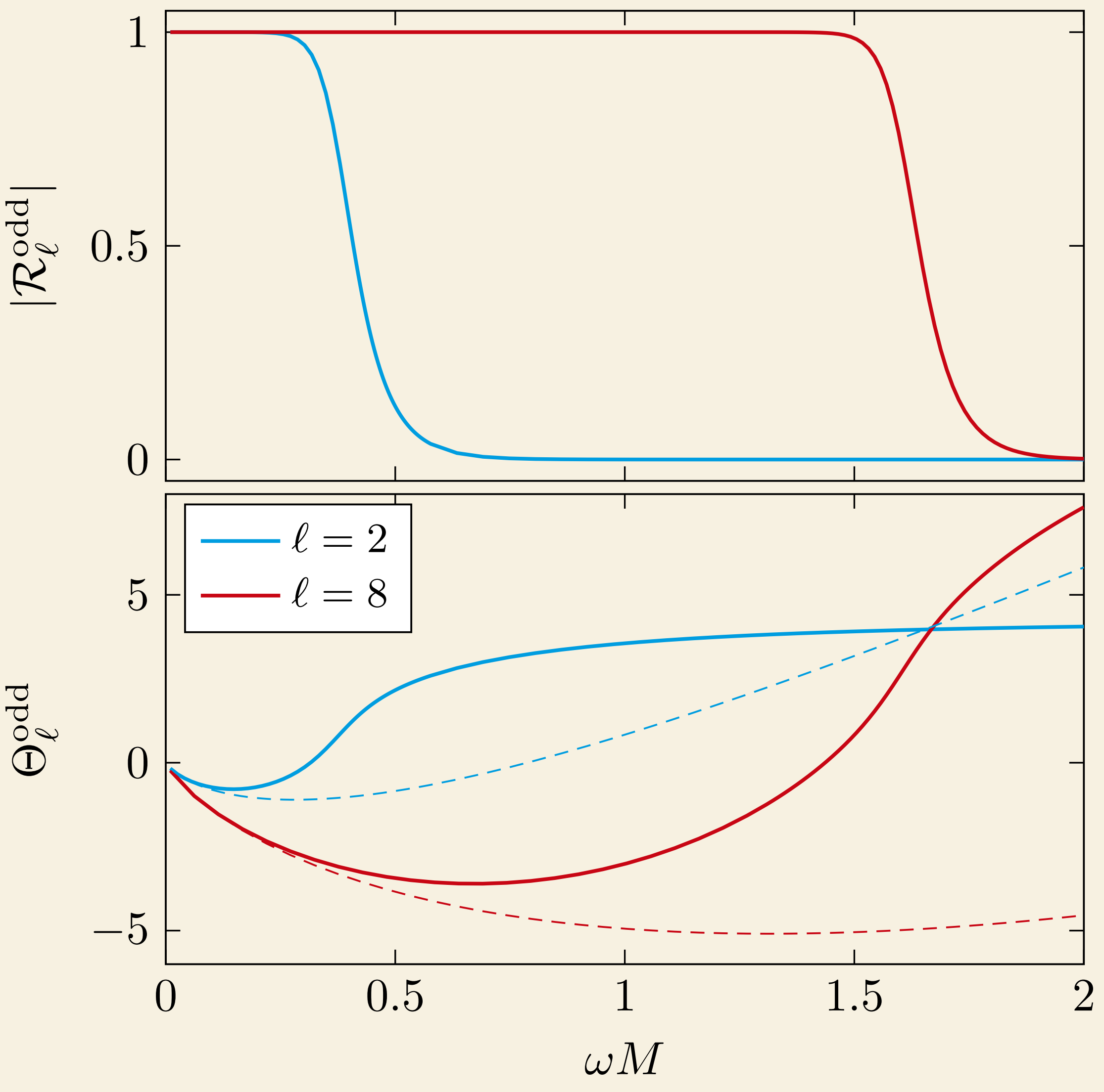
$$\psi_{\ell mp} = A_{\rm in}e^{-i\omega r} + A_{\rm out}e^{i\omega r}\,,\, r \to \infty$$

$$\mathscr{R}_{\ell p}(\omega M) = \frac{A_{\rm out}}{A_{\rm in}}$$

Black Hole Reflectivities!

*[Regge & Wheeler (1957), Zerilli (1970)]*

# Scattering of GWs by a BH




Can be computed using different techniques…

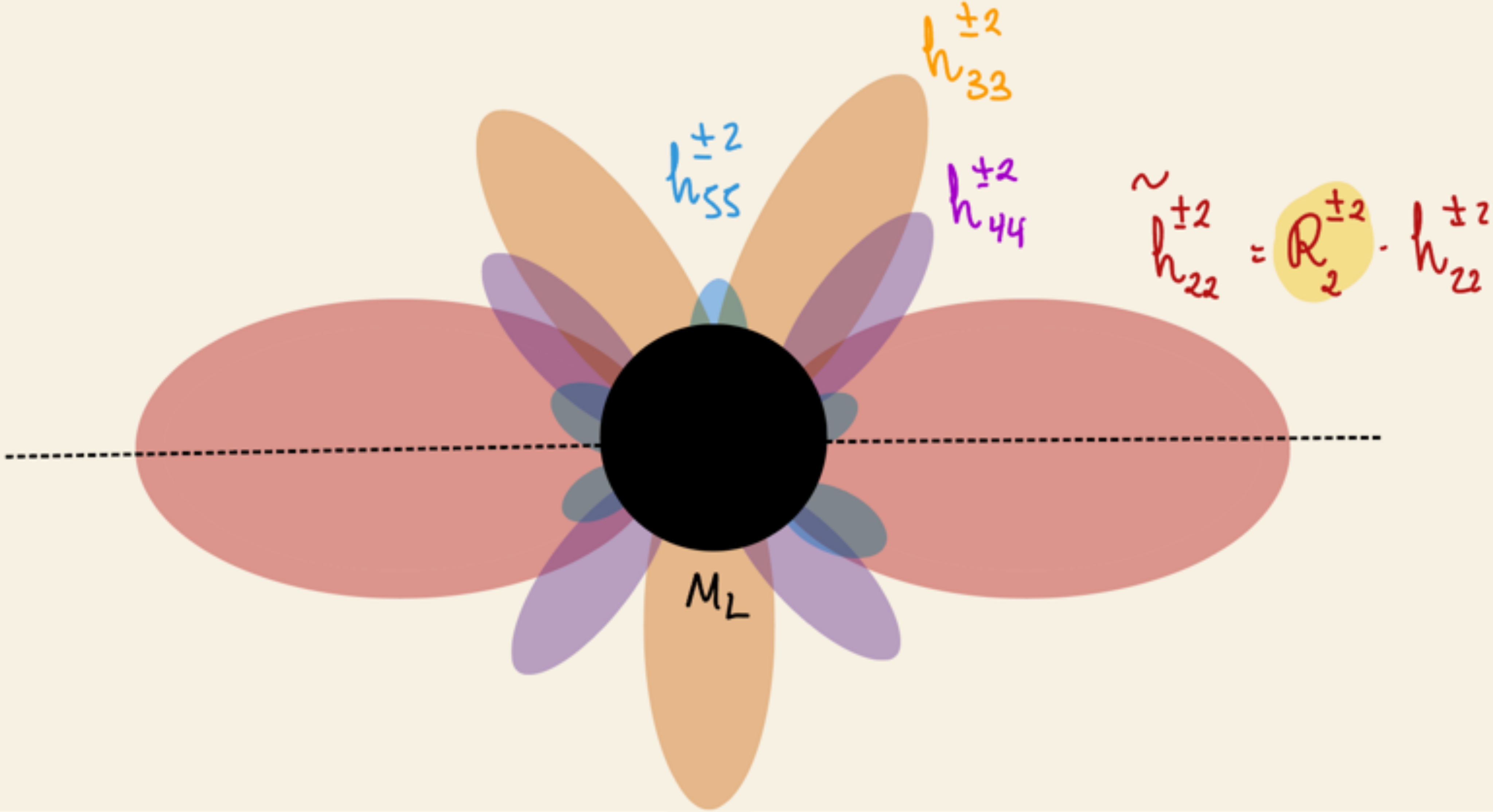
- Direct Integration
- Mano-Suzuki-Takasugi
- Scattering Amplitudes (low $G\omega M \ll 1$ )
- WKB or Eikonal (high $\omega M \gg 1$)



[Carrillo Gonzalez + (2026)]

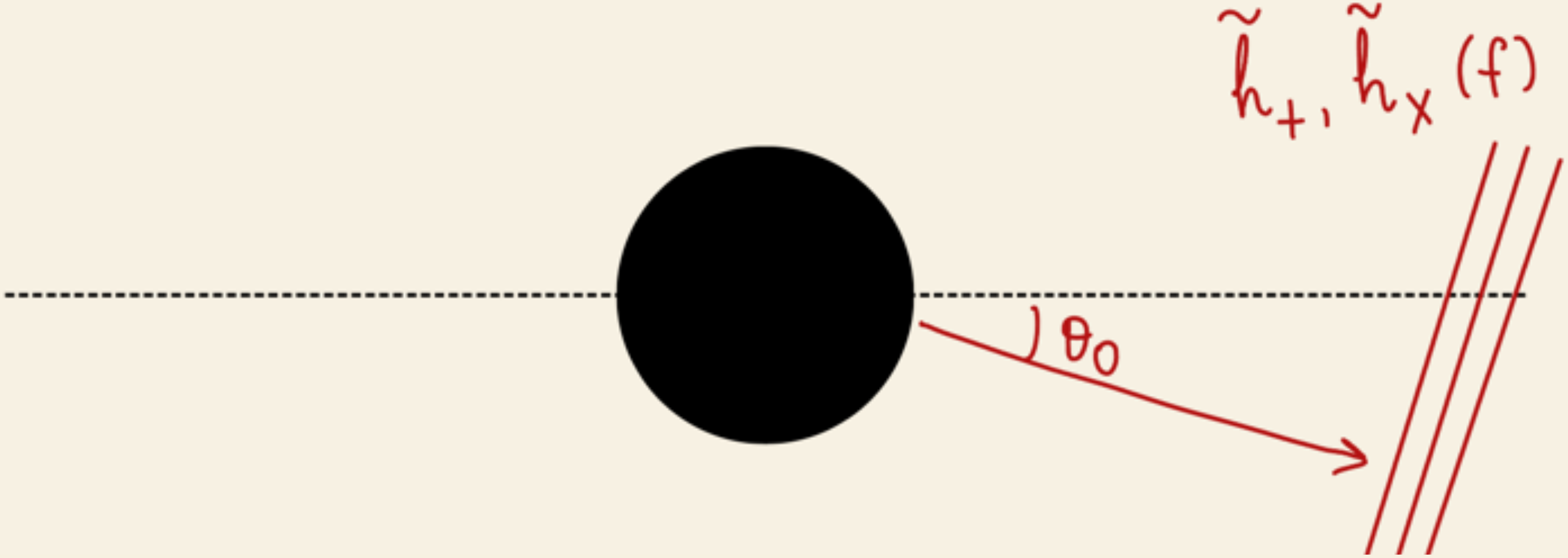
# BHPT | Lensing





# BHPT | Lensing





# BH Amplification Factor

Putting pieces together, the amplification factor now mixes polarizations!

$$\tilde{h}_{+,\times} = F h_{+,\times} + G h_{\times,+}$$

[mixing from a Schwarzschild BH is just a phase artifact, though]

$$F = 1 + [\mathrm{geom.}] \sum_{\ell, |\mathrm{m}| = \pm 2} \sqrt{2\ell + 1}\, \mathrm{Y}^{(\pm 2)}_{\ell \mathrm{m}}(\Omega_{\mathrm{OL}}) \Big[ 1 + (-1)^{\ell} \mathscr{R}_{\ell, \mathrm{sign(m)}}(\mathrm{f}) \Big]$$

There is a small problem — the sum diverges

# Crisis

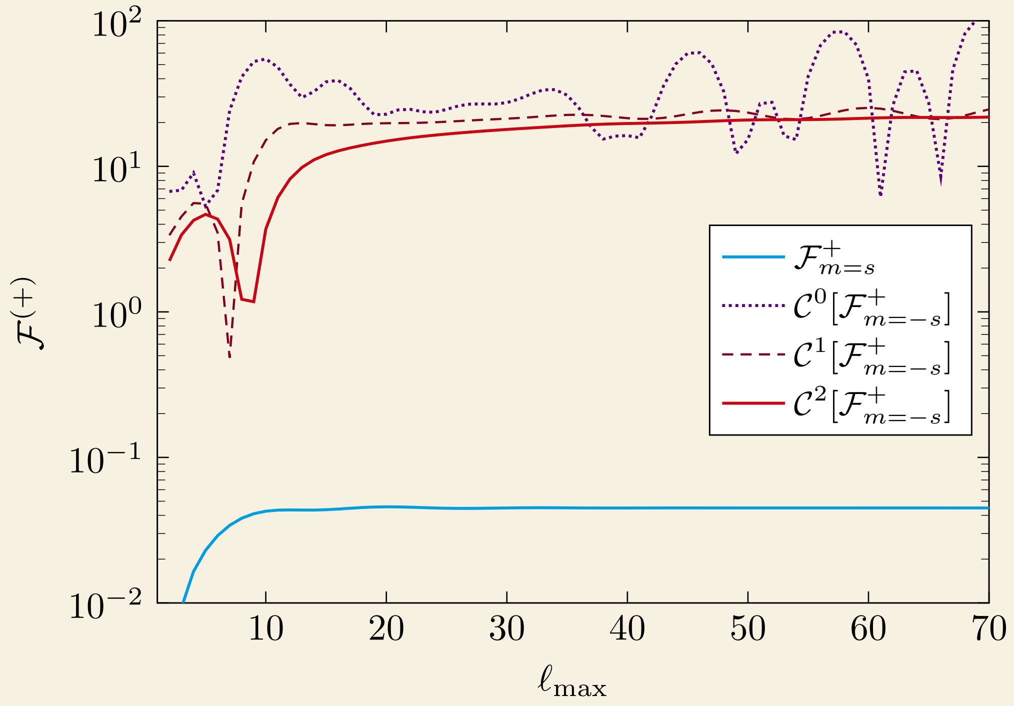


# Cèsaro Summation



A (mathy) way of summing a divergent series, like 1 + 2 + 3 + … = -1/12

$$\mathscr{C}^\alpha[z_n] = \lim_{N\to\infty} \sum_{n=0}^{N} \binom{N}{n} \binom{N+\alpha}{n}^{-1} z_n$$

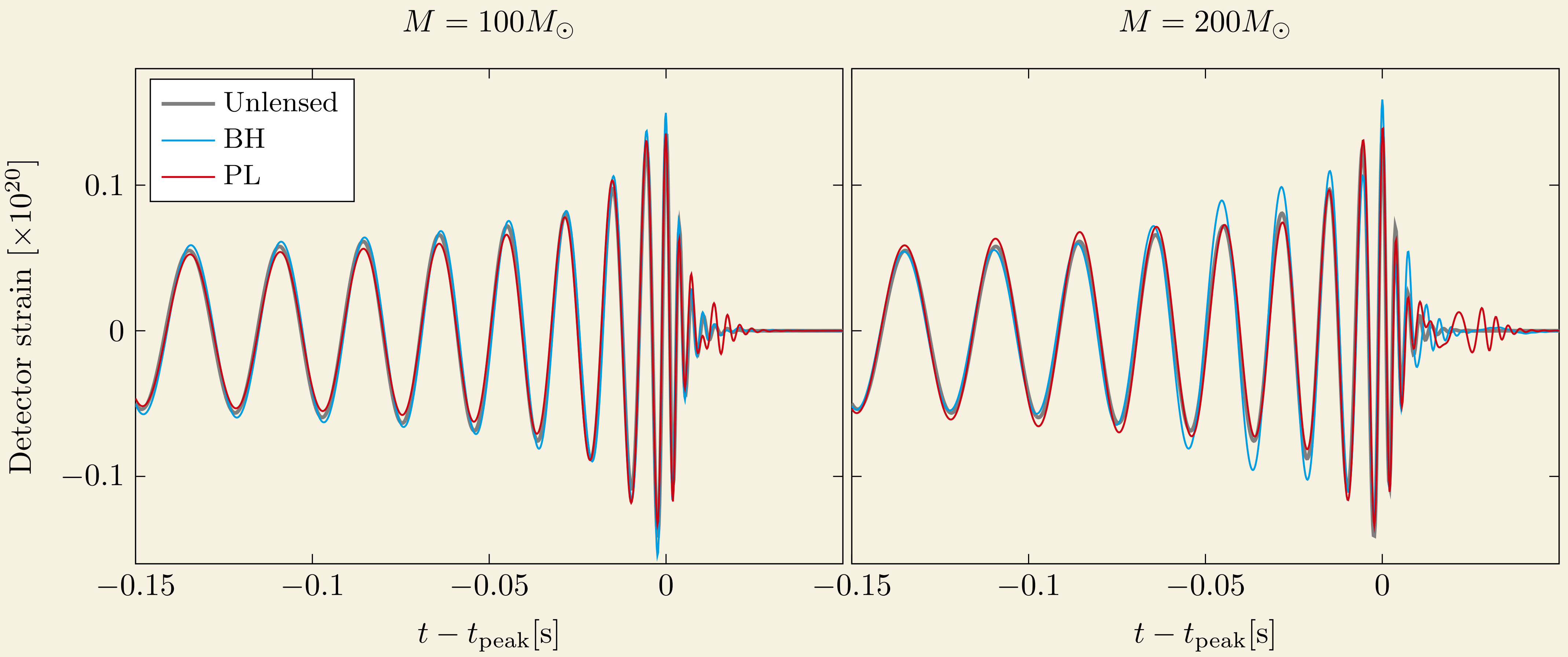


# Crisis Averted

# Are BHs Point Lenses?



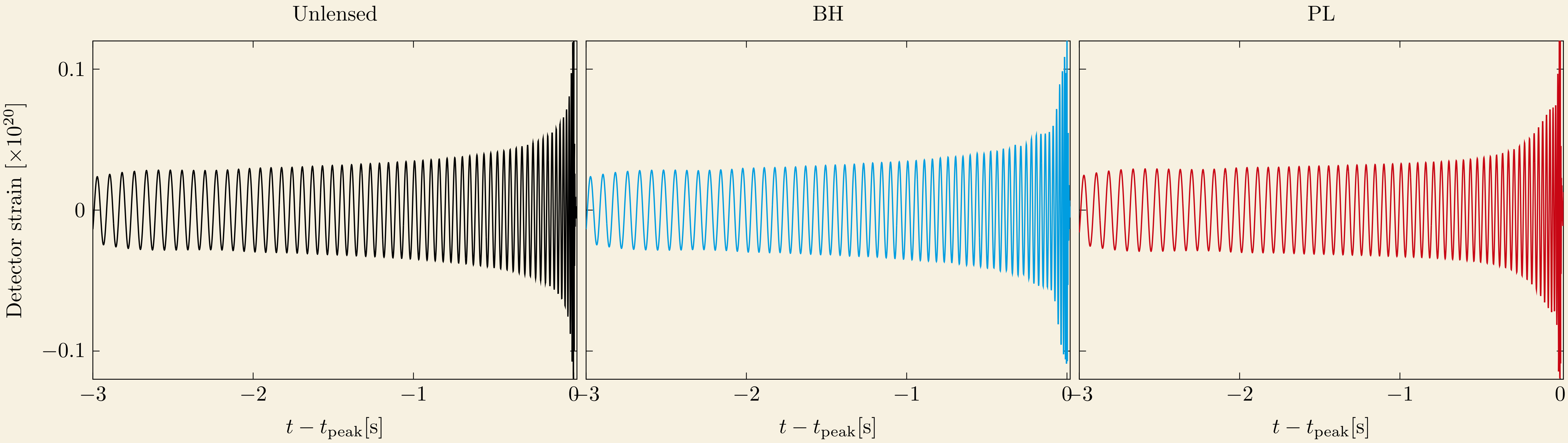
System: 30 + 30 $M_\odot$ BBH at $z = 0.1$, with $y = 2.69$ [convergence], $w \sim 0.6 - 3.6$



# Are BHs Point Lenses?



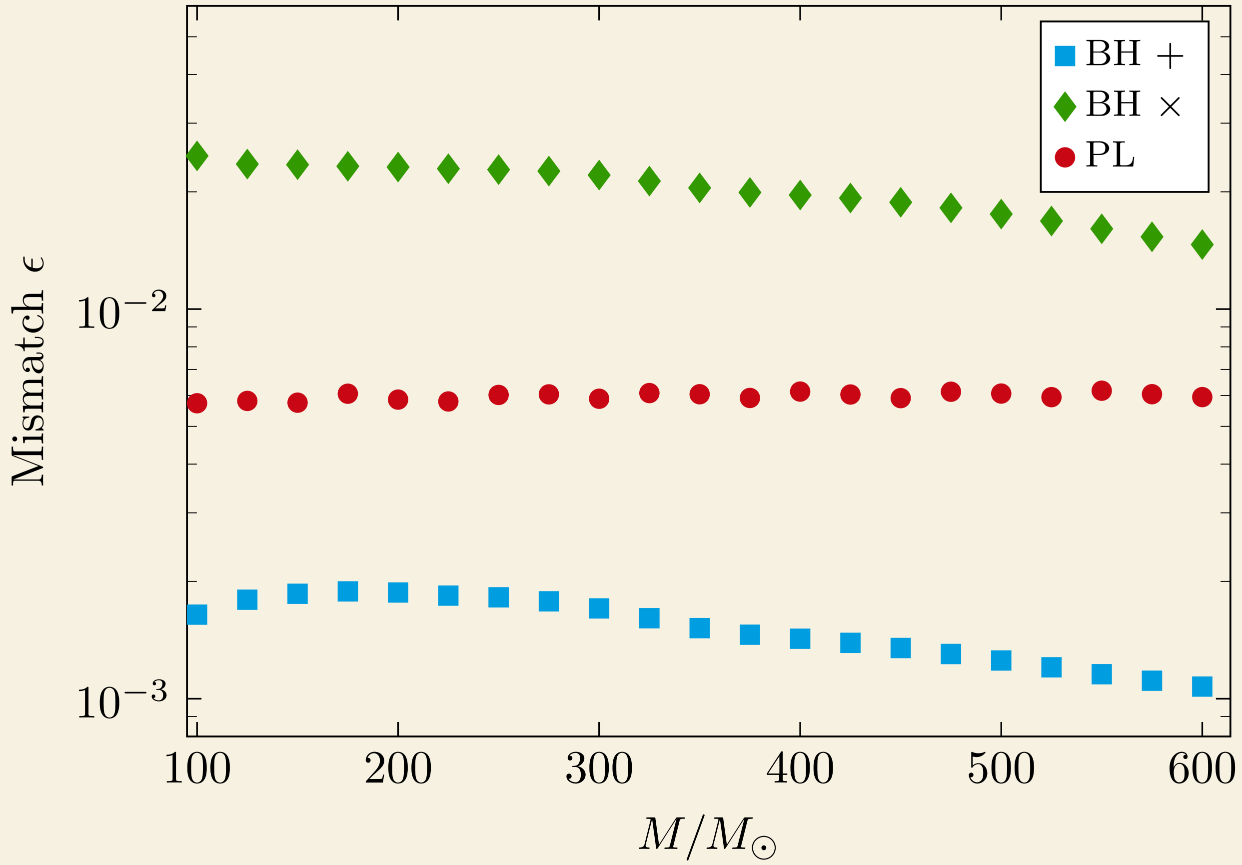
System: $30 + 30\ M_\odot$ BBH at $z = 0.1$, with $y = 2.69$ [convergence], $w \sim 0.6 - 3.6$



# Are BHs Point Lenses?



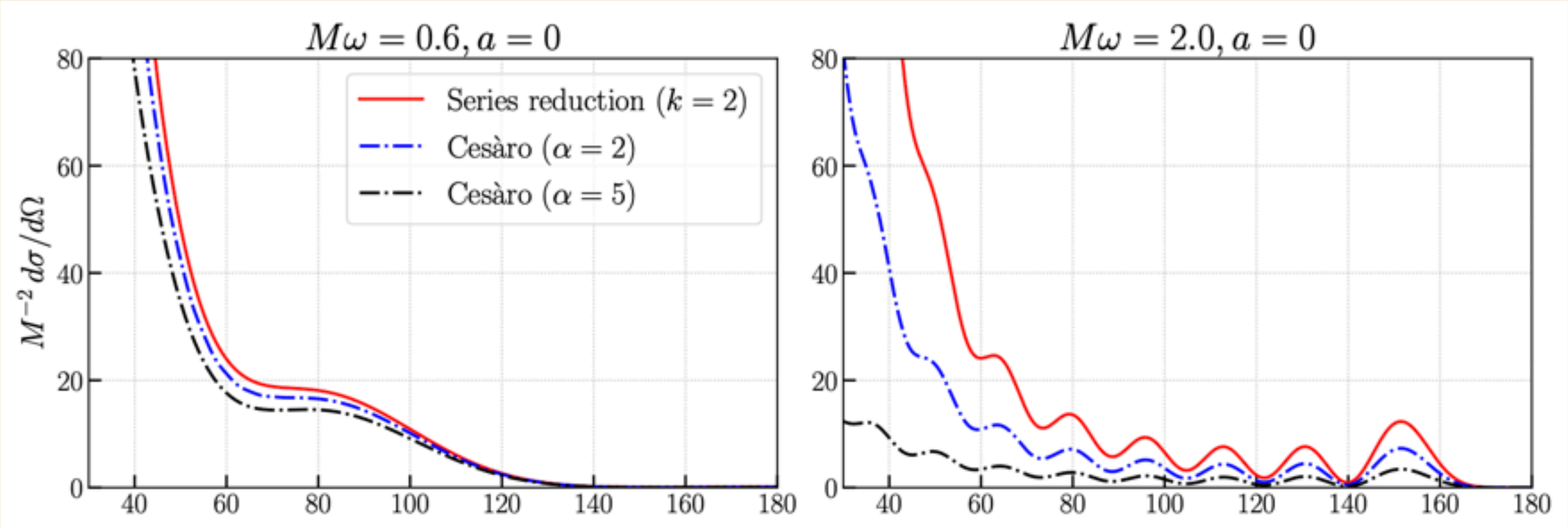
Small differences — but not trivial, even at large y. Important for PE!



# New Developments



Actually, the Cèsaro summation had problems at high frequencies…




Series Reduction method works! And now, framework is extended to Kerr.



*[Kubota+ (2024), Saketh,Ghosh,Mishra (2025)]*

# Take-Aways



### What we showed

- ✓ How to translate Lensing — BHPT
- ✓ In the strong field: amplification "matrix"
- ✓ BHs are not too different from point masses
- ✓ Difficulty is in the plane wave > partial wave decomposition

### Open questions

- Push calculations to small impact parameter!
- GO limit from BHPT?
- Astrophysical probablity of lensing by an isolated BH vs DM Halo?

# III Mergers in Triple Systems

*[based on 2605.20320]*
*in collaboration with*

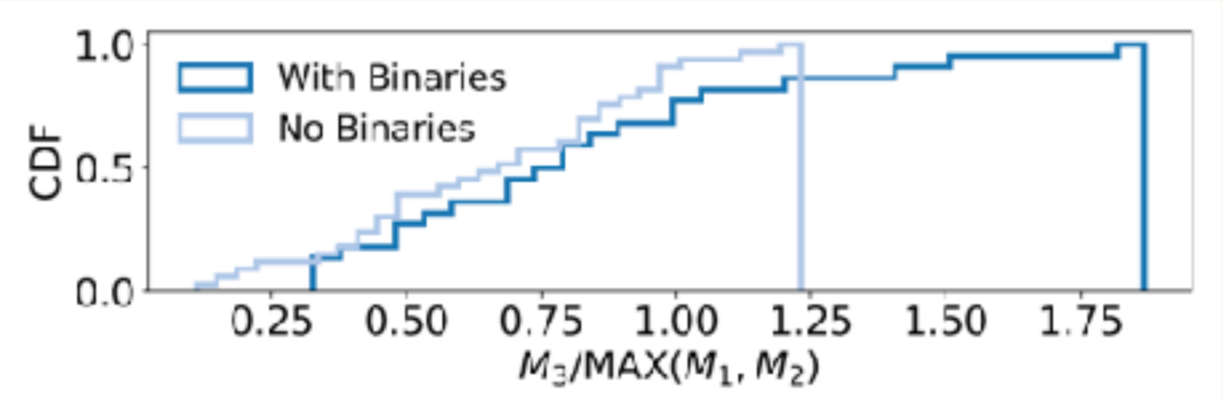


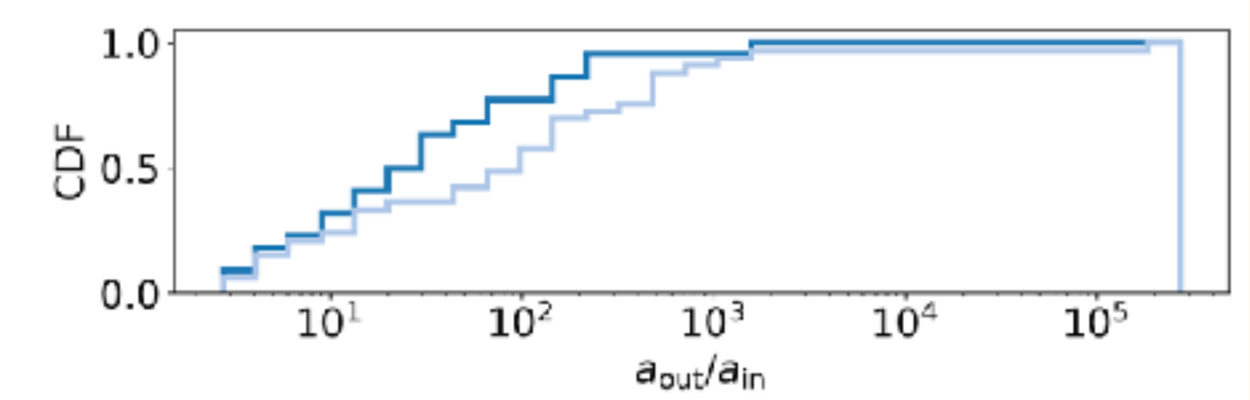
Vitor Cardoso
(NBI/IST-Lisbon)

Giuseppe Ficarra
(U Calabria)

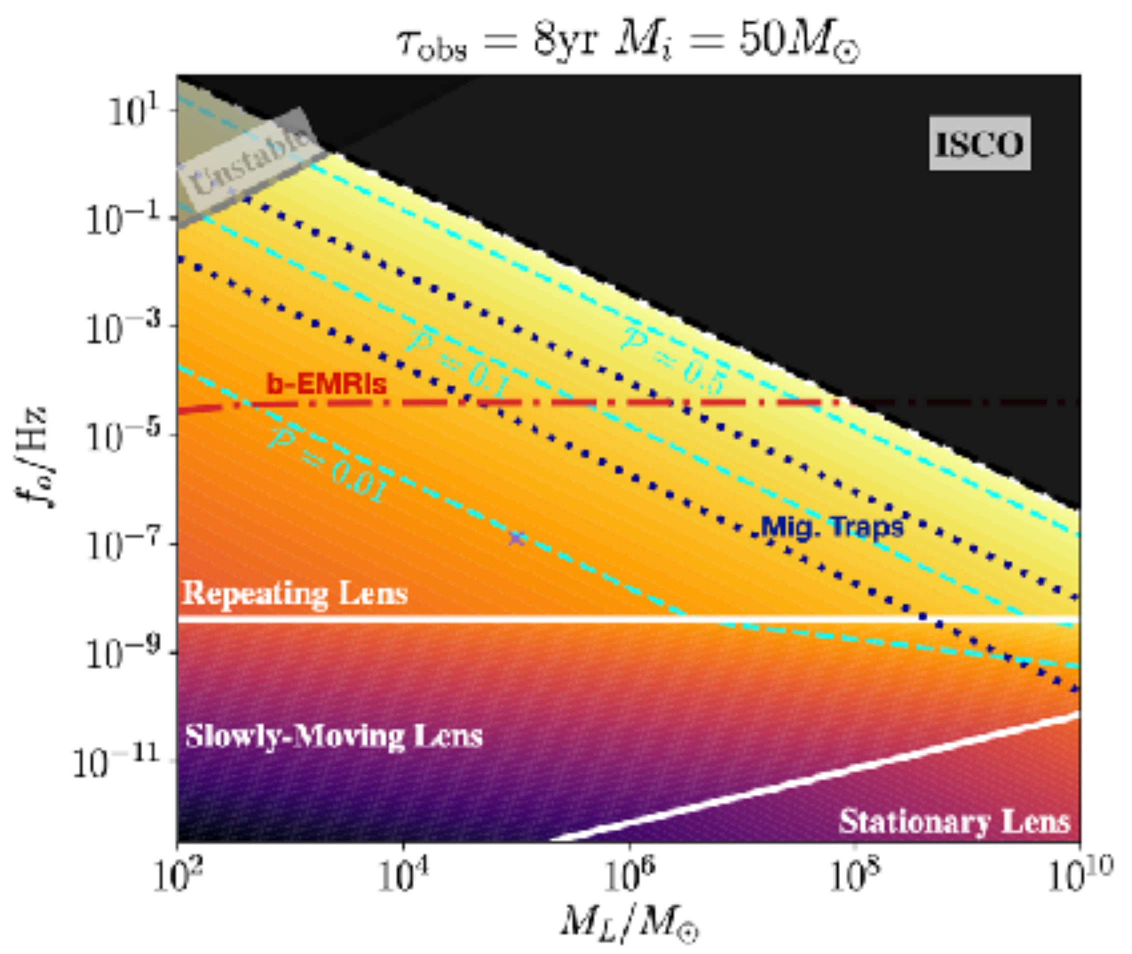
João Santos
(IST-Lisbon)

# The Universe is a Factory of Triples



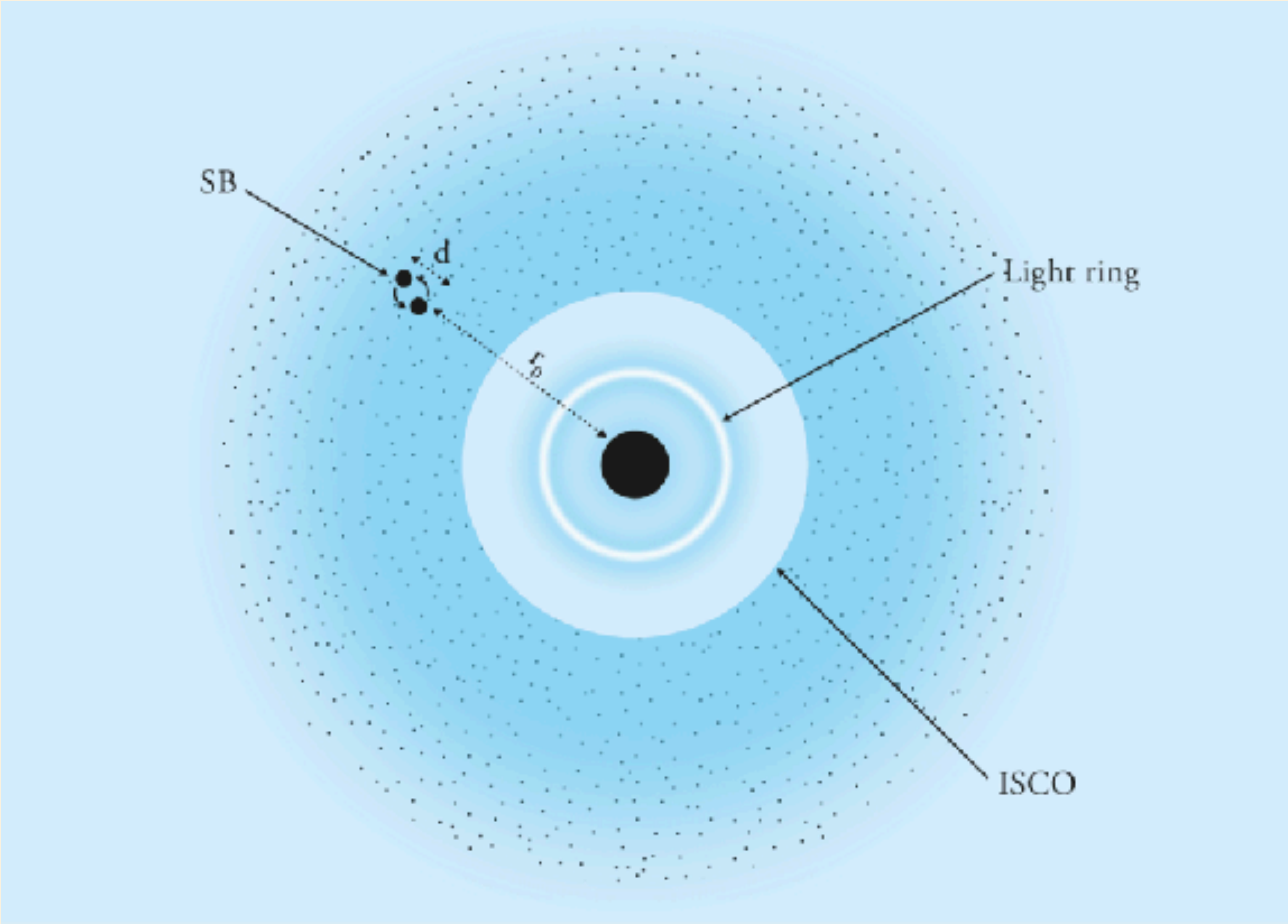

[Barber & Antonini (2025)]




[D'Orazio & Loeb (2018)]

# bEMRIs








[Santos+ (2025, 26)]

# The Question



How do the GWs emitted during the merger in a triple system look like?

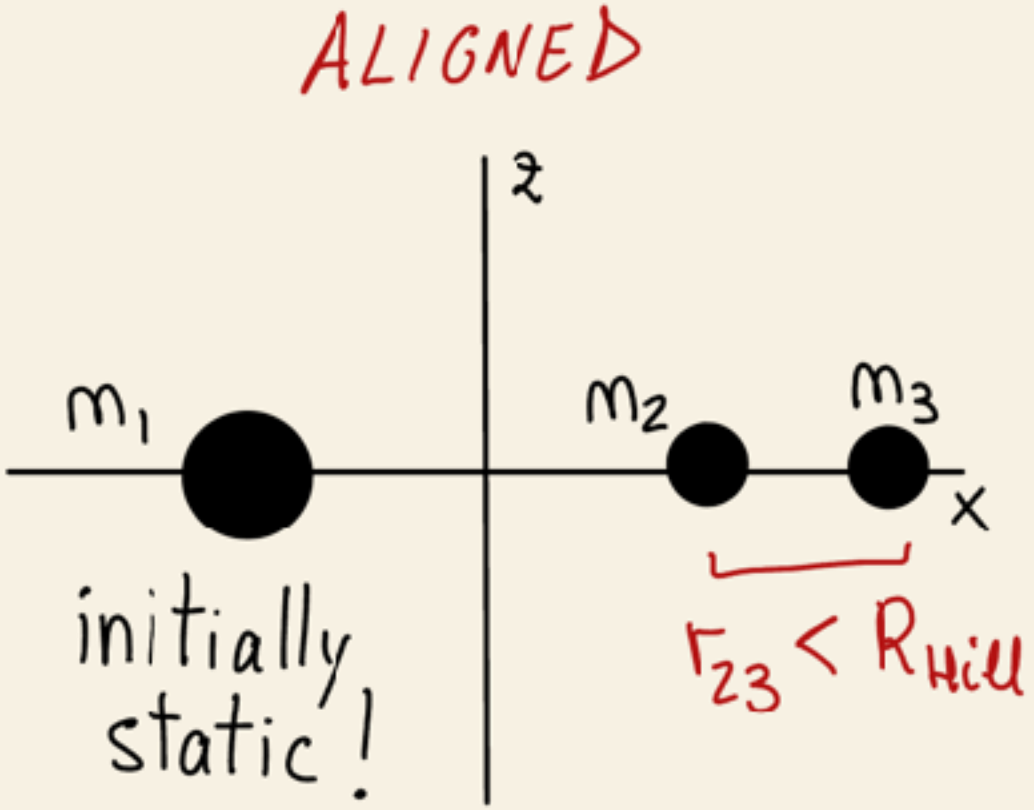
No simplifying assumptions: comparable masses, small distances.

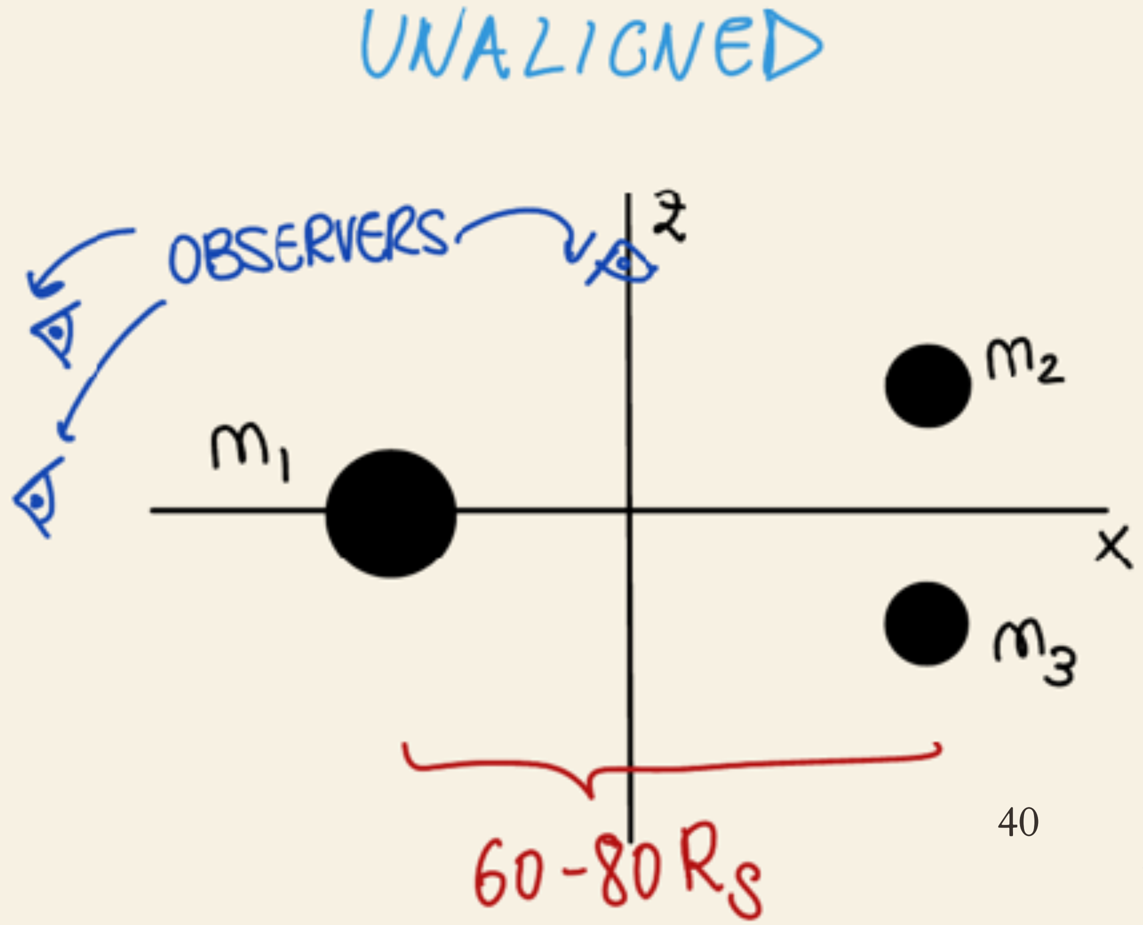
What's the craziest thing that can happen?

# Set-Up

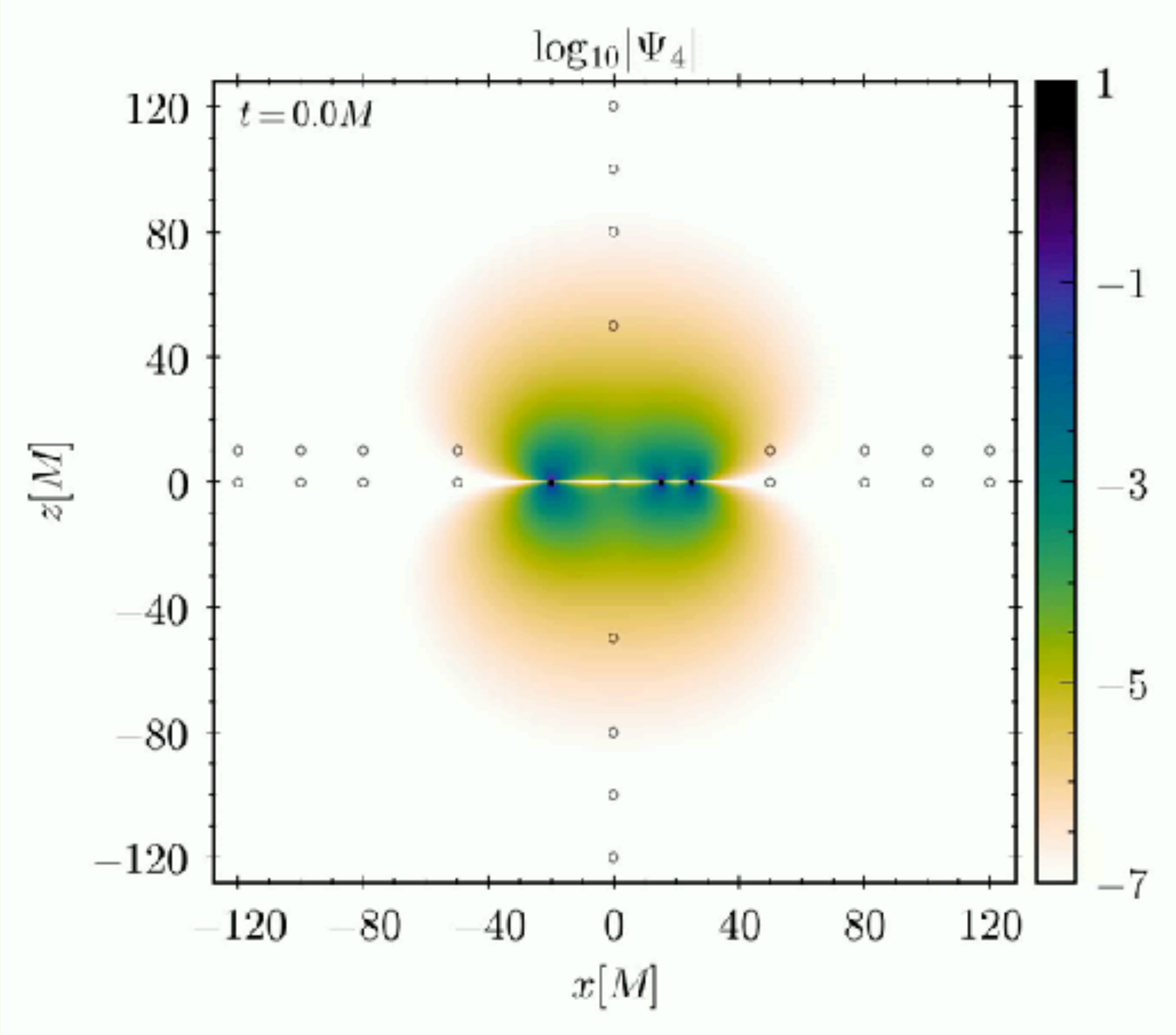
Numbers are driven by what Numerical Relativity can do in a reasonable time

(~1 week / high-resolution run)



# Set-Up: Aligned



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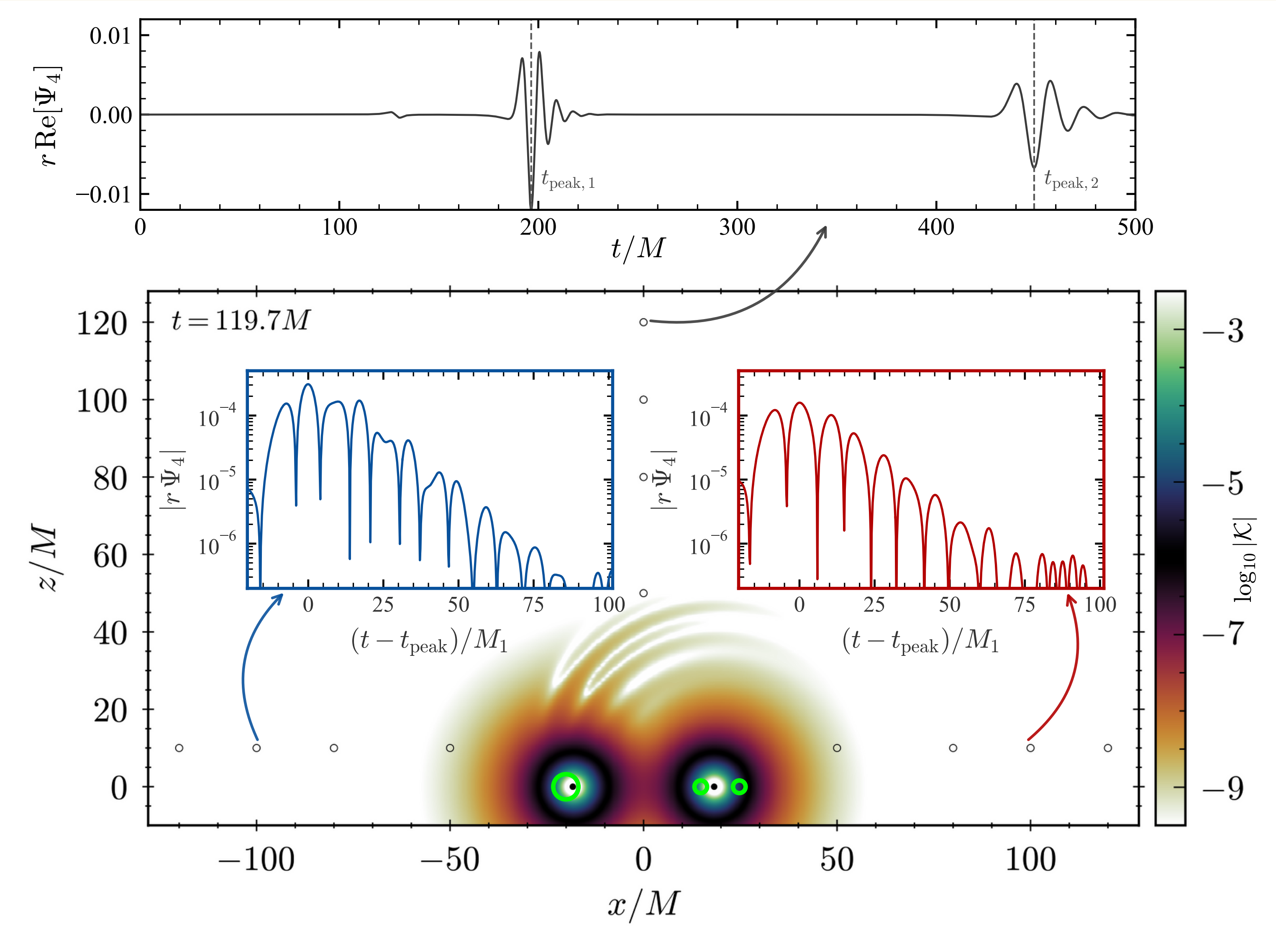


# Analysis of Merger-Ringdown



The merger-ringdown is governed by a superposition of the BH QNMs:

$$h \approx \sum_{\ell mn} A_{\ell mn} e^{-i\omega_{\ell mn}(t-t_{\rm mg})} {}_{-2}S_{\ell mn}(\theta, \phi)$$

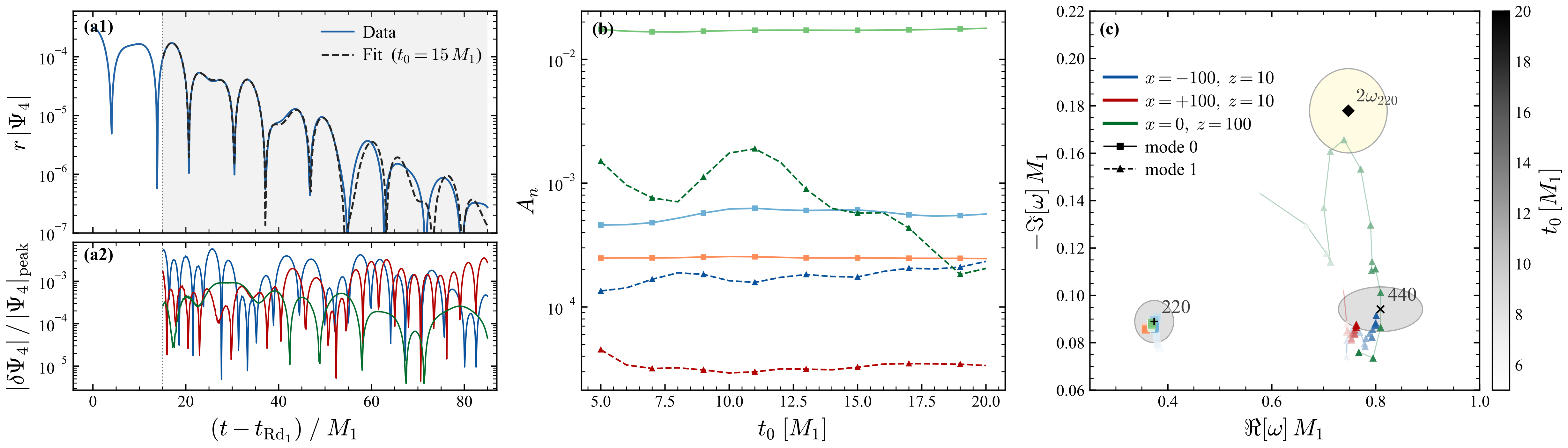
We use `jaxqualin` to extract the amplitudes and frequencies of the dominant modes

The frequencies should be: (i) gravitationally redshifted and (ii) Doppler shifted

The amplitudes should be: (i) (de)magnified by GW lensing and (ii) resonantly excited

# Analysis of Merger-Ringdown






It's hard to resolve more than 2 modes in a stable way! Even "without" noise

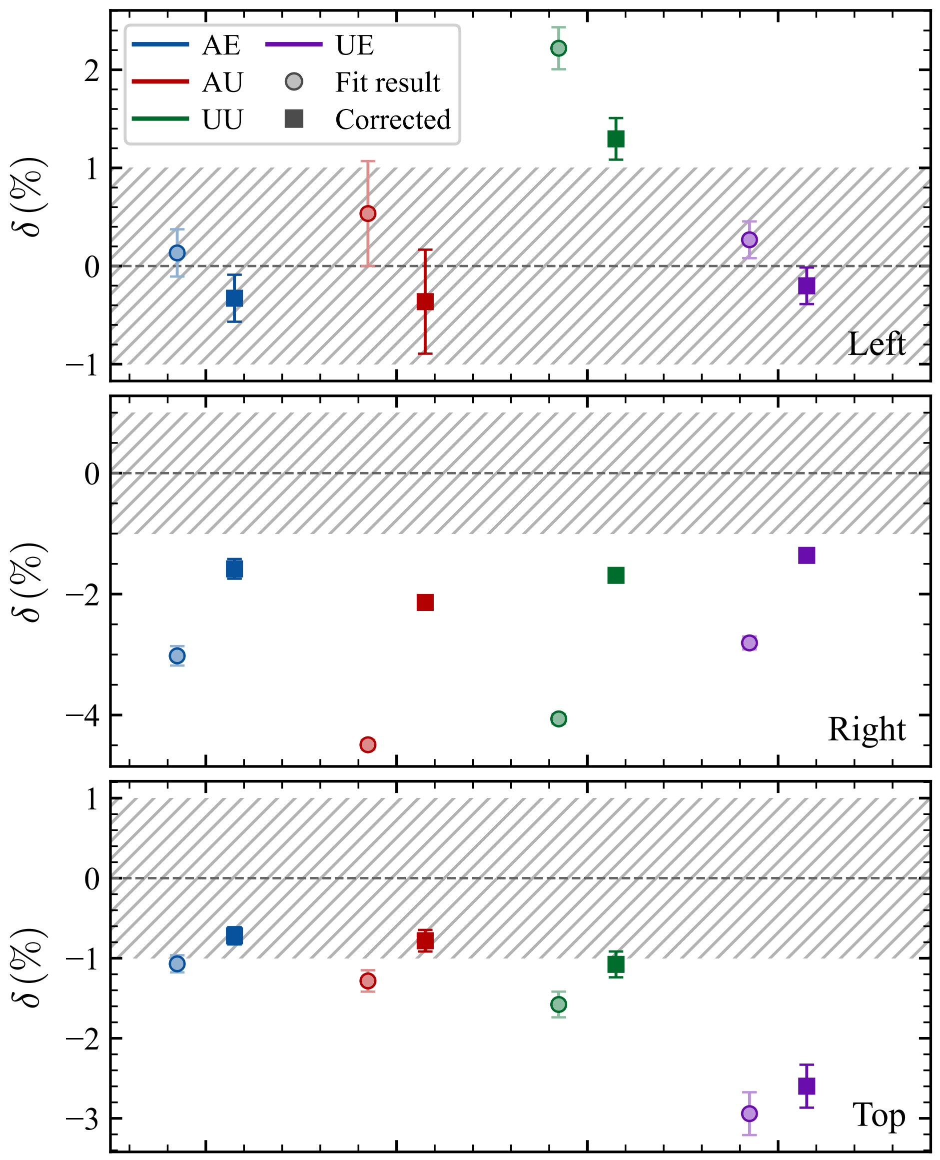
# Frequencies: Redshift & Doppler

Correct the extracted "dominant" frequency accounting for redshift effects

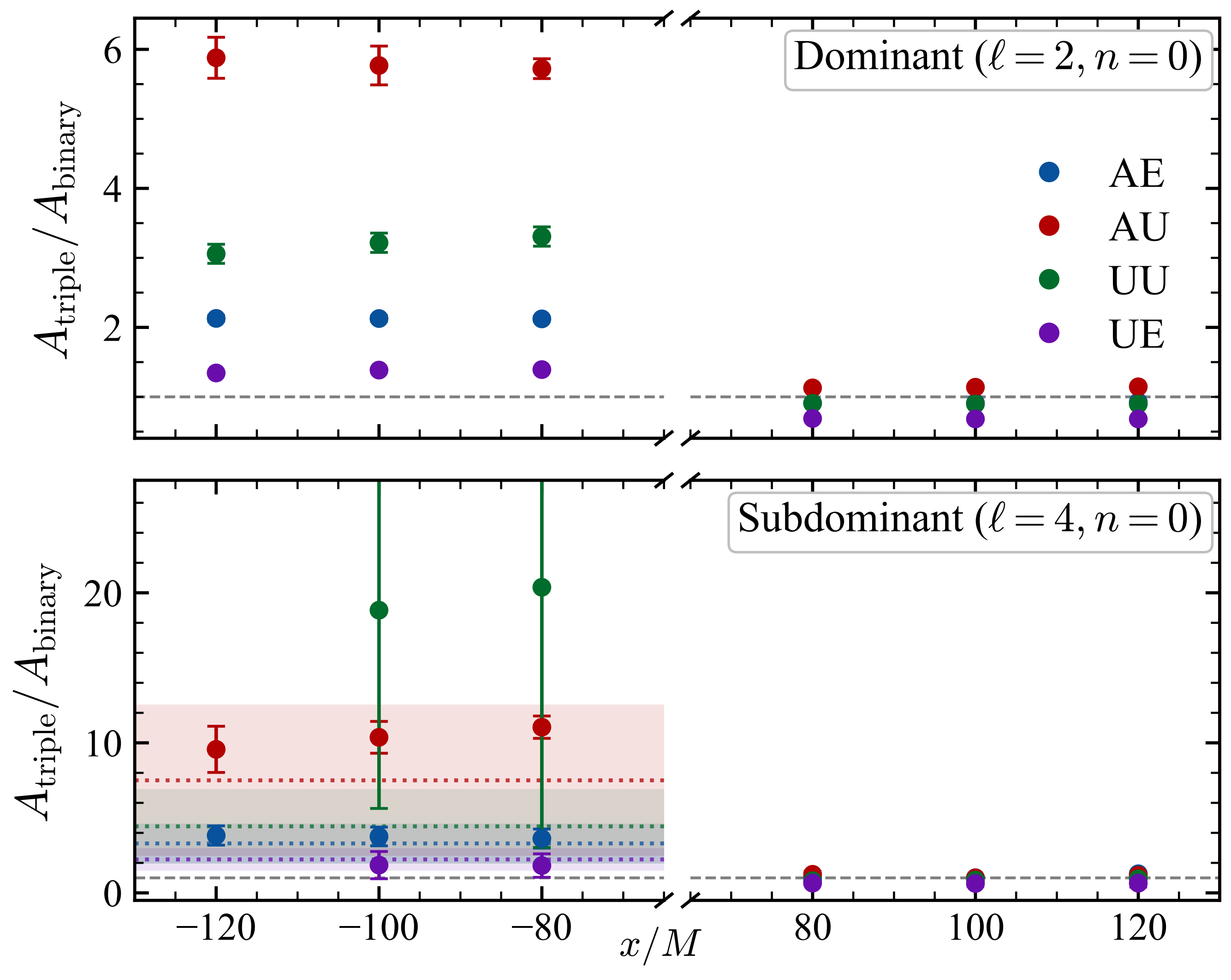
$$f_R = \sqrt{\frac{1 - 2m_1/r_{\mathrm{obs}}}{1 - 2m_1/r_{\mathrm{mg}}}}[\gamma(1 - v\cos\theta)]^{-1} f_{200}$$

Check which residual is better:

$$\delta = \frac{f - f_{\mathrm{th}}}{f_{\mathrm{th}}}$$



# Amplitudes: Amplification




We run simulations with no companion to understand the effects of lensing

For observers to the right: no effect!

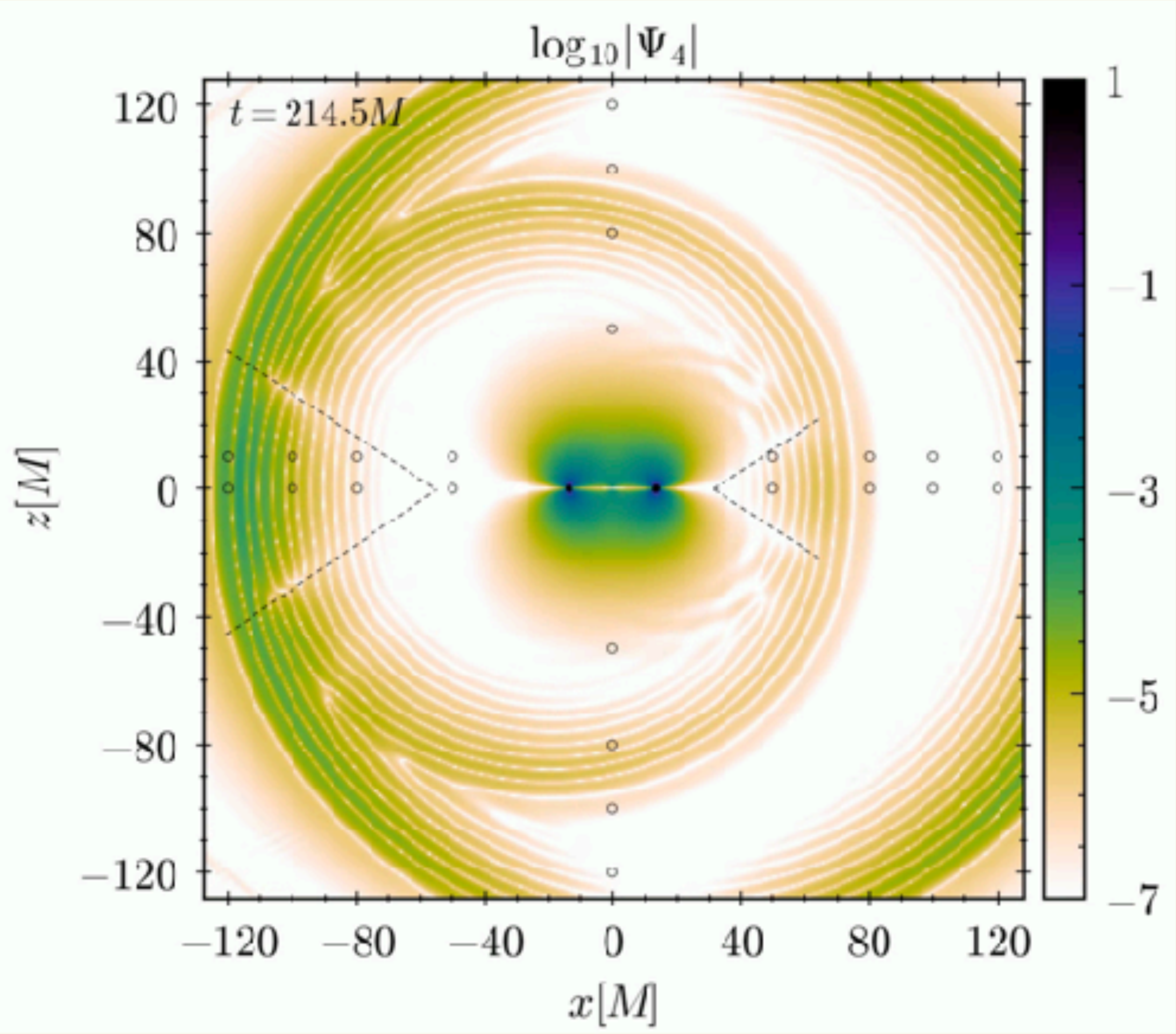
Observers to the left see larger amplitudes, with a frequency-dependent magnification!

$$F \sim [\text{geom.}] \mathrm{M_L} \omega$$

# Wave Optics

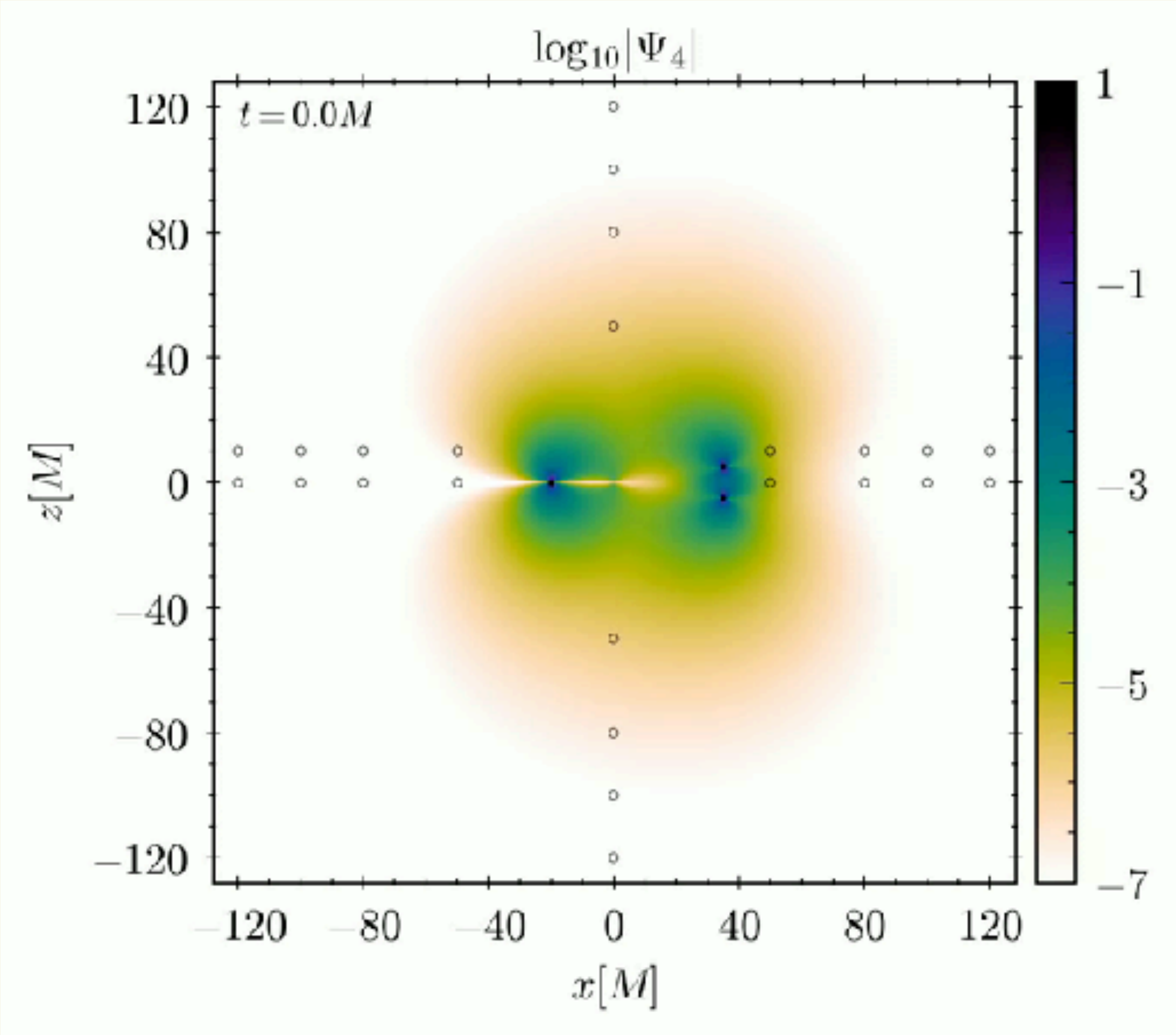
Intereference fringes, at an angle scaling that can be* predicted.

Further developments in this direction will follow!



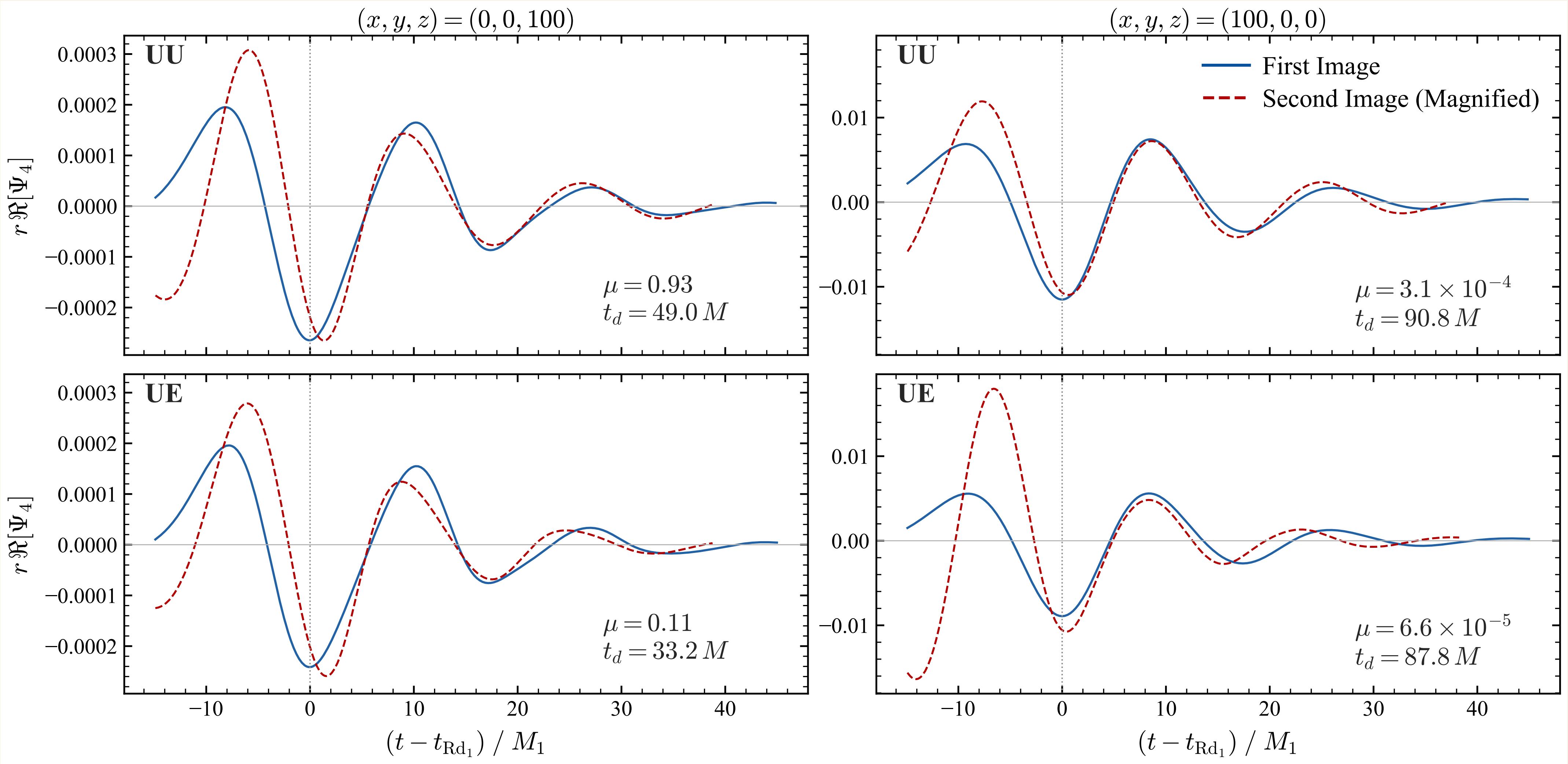

[Santos & Vujeva (2026, to appear)]

# Set-Up: Un-Aligned



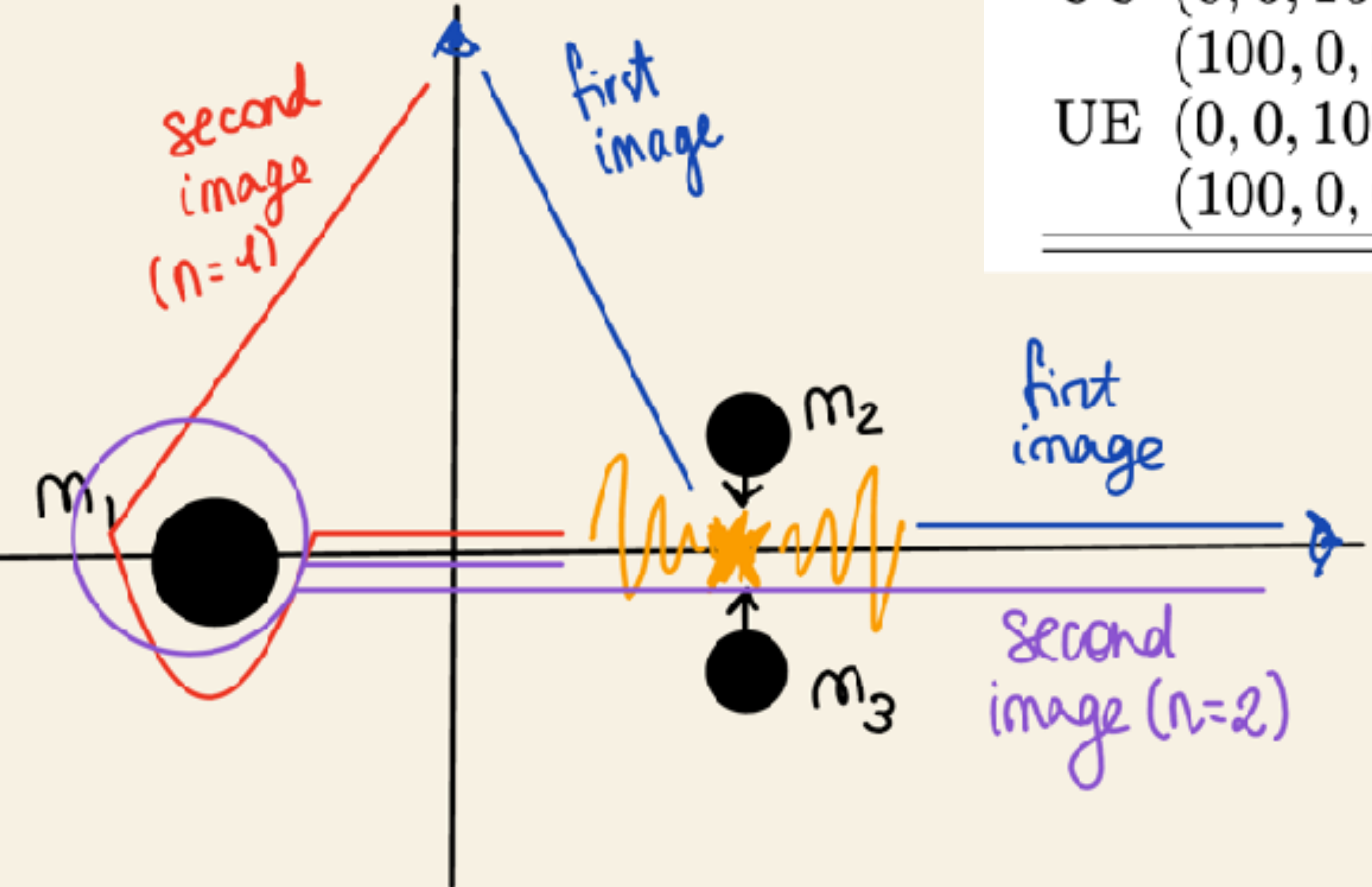
# Secondary Images





# Secondary Images

| ID | Observer | $\Delta t_{\rm num}/M$ | $\Delta t_{\rm go}/M$ | $t_d/M$ |
|---|---|---|---|---|
| UU | $(0,0,100)$ | $52 \pm 8$ | 52 | 49.0 |
| | $(100,0,0)$ | $95 \pm 8$ | 107 | 90.8 |
| UE | $(0,0,100)$ | $37 \pm 11$ | 50 | 33.2 |
| | $(100,0,0)$ | $92 \pm 11$ | 103 | 87.8 |



# Take-Aways

## What we showed

- ✓ NR can do triple systems!
- ✓ GWs emitted are
  - Doppler shifted
  - Gravitational redshifted
  - Lensed (no WO nor GO works well)
  - Some evidence for resonances
- ✓ Ringdown Analysis can capture this!

## Open questions

- What happens when there's an inspiral? We expect more resonances
- What if there is mass asymmetry?
- Are (perturbative) bEMRI models accurate if $q \ll 1$

ありがとうございます